DECOMPOSITION OF TORSION THEORIES

By

RICHARD BRONOWITZ

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DECOMPOSITION OF TORSION THEORIES

By

Richard Bronowitz

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Chairman: Mark L. Teply
Major Department: Department of Mathematics

The purpose of this paper is to extend the concept of Primary Decomposition of modules in the simple torsion class to modules in an arbitrary torsion class. The main theorem gives necessary and sufficient conditions on a certain subclass of cyclic torsion modules for the torsion class to admit a decomposition. In order to obtain this, results concerning torsion preradicals and the lattice of torsion classes are proved. Results of Dickson, Alin and Albu are obtained as corollaries to the main theorem. Finally, those rings, for which all torsion classes are central splitting, are determined to be finite direct sums of full matrix rings over local semi-artinian rings.

CHAPTER 1

Introduction

An element of an Abelian group is said to be a torsion element if it has finite order. Since Abelian groups may be regarded as modules over the integers, the definition may be stated in module-theoretic terms. That is to say, an element of an Abelian group is a torsion element if it has a non-zero annihilator. An Abelian group is called a torsion Abelian group if each of its elements is a torsion element. Then any Abelian group has a maximal torsion subgroup; namely the subgroup consisting of all the torsion elements of the group.

This concept is easily generalized to modules over any commutative integral domain. However, when one moves to an even slightly more general ring, the collection of elements which have non-zero annihilators does not form a submodule. An example of this occurs when the ring under consideration is the direct product of two fields. However, there is at least one reasonable way of generalizing the concept of torsion.

This generalization was introduced by Dickson in 1966 [5]. His definition holds for certain Abelian categories. For the purposes of this dissertation, the definition will be restricted to R-mod, the category of left unital R-modules, where R is any ring with identity. All modules in this work will be members of R-mod. According to Dickson, a *torsion theory* is a pair, $(\mathfrak{T},\mathfrak{F})$, of classes of R-modules satisfying the following axioms:

(1) $\mathfrak{T} \cap \mathfrak{F} = \{0\}$;

(2) $\mathfrak{T}$ is closed under homomorphic images;

(3) $\mathfrak{F}$ is closed under submodules;

(4) For all R-modules, M, there is a submodule, $\mathfrak{T}(M)$, of M such that $\mathfrak{T}(M) \in \mathfrak{T}$ and $M/\mathfrak{T}(M) \in \mathfrak{F}$.

A class of R-modules, $\mathfrak{T}$, is called a *torsion* (*torsion-free*) class if it is the first (second) coordinate of some torsion theory. A class of R-modules, $\mathfrak{T}$, is a *torsion class* if $\mathfrak{T}$ is closed under direct sums, homomorphic images, and extensions of one module in $\mathfrak{T}$ by another. Recall that $\mathfrak{T}$ is closed under extensions if for all exact sequences,

$$0 \to A \to B \to C \to 0,$$

with A and C in $\mathfrak{T}$, then B is also in $\mathfrak{T}$. Dually, a class of modules, $\mathfrak{F}$, is a torsion-free class if it is closed under submodules, direct products, and extensions of one module in $\mathfrak{F}$ by another. In a torsion theory, $(\mathfrak{T},\mathfrak{F})$, the modules in $\mathfrak{T}$ are called *torsion*, and the modules in $\mathfrak{F}$ are called *torsion-free*.

Once a torsion class, $\mathfrak{T}$, is specified, the torsion theory having $\mathfrak{T}$ as its torsion class is uniquely determined. In this situation, $\mathfrak{F} = \{M \in \text{R-mod} \mid \text{Hom}(T,M) = 0 \text{ for all } T \in \mathfrak{T}\}$. Similarly, if a torsion-free class, $\mathfrak{F}$, is specified, then the torsion theory having $\mathfrak{F}$ as its torsion-free class is uniquely determined. In this instance, $\mathfrak{T} = \{M \in R\text{-mod} \mid \mathrm{Hom}(M,F) = 0$ for all $F \in \mathfrak{F}\}$. The above results are due to Dickson [5].

A torsion theory is called hereditary if its torsion class is closed under submodules. In this event, the torsion class is called hereditary. This concept has appeared in the literature under such names as torsion radical [10], idempotent radical [15], and strongly complete Serre class [20]. Dickson [5] shows that a torsion class is hereditary if and only if the corresponding torsion-free class is closed under injective envelopes. If M is an R-module, then the injective envelope of M, written E(M), is a minimal injective extension of M [9]. Unless otherwise mentioned all torsion theories will be hereditary, and the term "hereditary" will be omitted.

Definition 1.1 [11]. A collection of left ideals of R, $F(\mathfrak{T})$, is called a topologizing and idempotent filter if it satisfies the following conditions:

(1) If $I \in F(\mathfrak{T})$ and $I \subseteq J$, then $J \in F(\mathfrak{T})$;

(2) If $I \in F(\mathfrak{T})$ and $J \in F(\mathfrak{T})$, then $I \cap J \in F(\mathfrak{T})$;

(3) If $I \in F(\mathfrak{T})$ and $r \in R$, then $(I:r) = \{x \in R \mid xr \in I\} \in F(\mathfrak{T})$;

(4) If $(J:r) \in F(\mathfrak{T})$ for all $r \in I$ and $I \in F(\mathfrak{T})$, then $J \in F(\mathfrak{T})$.

Gabriel then proved that there is a 1-1 correspondence between hereditary torsion classes, $\mathfrak{T}$, and topologizing and idempotent filters, $F(\mathfrak{T})$, given by the mapping

$$\mathfrak{T} \to F(\mathfrak{T}) = \{I \subseteq R \mid R/I \in \mathfrak{T}\}.$$

These definitions can be made for classes of right R-modules and filters of right ideals in an entirely analogous manner.

It is in order to look at some examples of torsion theories. This will be done by specifying their torsion classes. By a previous remark, this means that the entire torsion theory is specified. Two torsion classes of a somewhat trivial nature are $\{0\}$ and R-mod. For the sake of completeness, the next example will be an example of a non-hereditary torsion class. If R is any ring with identity, then $\mathfrak{J} = \{M \in R\text{-mod} \mid \mathrm{Hom}(M,R) = 0\}$ is a torsion class, which, as in the case of Abelian groups, is not necessarily hereditary. A torsion class defined in a similar manner is Jans' E(R)-class [13]. In this theory, $\mathfrak{J} = \{M \mid \mathrm{Hom}(M,E(R)) = 0\}$ is the torsion class. This class is hereditary, because E(R) is an injective R-module.

Definition 1.2 [12]. If N is a submodule of M, then N is called an essential submodule of M if $N \cap N' \neq 0$ for all non-zero submodules, N', of M.

Following Goldie [12], define $Z(M) = \{m \in M \mid (0{:}m)$ is an essential left ideal of R$\}$. If π_1 is the natural map from M onto $M/Z(M)$, then define $Z_2(M) = \pi_1^{-1}(Z(M/Z(M)))$. Let $\mathfrak{G} = \{M \in R\text{-mod} \mid Z_2(M) = M\}$. $\mathfrak{G}$ is a torsion class, which is known as the Goldie torsion class. The filter corresponding to the Goldie torsion class is the smallest filter containing all the essential left ideals of R. The torsion-free modules are those modules, M, such that $Z(M) = 0$. If $Z(R) = 0$, then the Goldie torsion class is equal to the $E(R)$-torsion class.

The final examples are due to Dickson and are called the torsion theories of simple type [5]. If $\mathcal{C}$ is any collection of non-isomorphic simple R-modules, then $\mathcal{T}_{\mathcal{C}} = \{M \mid$ Every non-zero homomorphic image of M has a submodule isomorphic to some member of $\mathcal{C}\}$ is a torsion class. The filter for this torsion class is the smallest filter containing all the maximal left ideals, M, of R such that R/M is isomorphic to a member of $\mathcal{C}$. The torsion-free class is precisely the class of modules which have no submodules isomorphic to an element of $\mathcal{C}$.

There are two torsion classes of simple type which deserve special recognition. If $\mathcal{C} = \{S\}$, then $\mathcal{T}_{\mathcal{C}}$ is denoted by $\mathcal{T}_S$ and is called the S-primary torsion class. If $\mathcal{C}$ is a complete representative set of non-isomorphic simple R-modules, the torsion class is called the simple torsion class and is denoted by $\mathcal{S}$. The torsion-free modules in this torsion theory are precisely those modules with no simple submodules.

A ring, R, is said to satisfy the Primary Decomposition if $\mathcal{S}(M) = \Sigma_{S \in \mathcal{S}} \mathcal{T}_S(M)$ for all $M \in$ R-mod. In this case, the simple theory is said to have the Primary Decomposition. If $\mathcal{T}_{\mathcal{C}}$ is a torsion theory of simple type, then $\mathcal{T}_{\mathcal{C}}$ is said to have the Primary Decomposition if $\mathcal{T}_{\mathcal{C}}(M) = \Sigma_{S \in \mathcal{C}} \mathcal{T}_S(M)$, for all $M \in$ R-mod. The question of when the simple theory has the Primary Decomposition has been investigated in recent papers; e.g. [2], [6], [7], and [18].

The purpose of this dissertation is to generalize the Primary Decomposition in the following way. Let $\mathfrak{J}$ be an arbitrary torsion class, and let $\{\mathfrak{J}_\alpha\}_{\alpha\in\mathfrak{a}}$ be a collection of subtorsion classes of $\mathfrak{J}$, i.e. $\mathfrak{J}_\alpha \subseteq \mathfrak{J}$ and $\mathfrak{J}_\alpha$ is a torsion class for all $\alpha \in \mathfrak{a}$. When is it true that $\mathfrak{J}(M) = \bigoplus_{\alpha\in\mathfrak{a}} \mathfrak{J}_\alpha(M)$ for all $M \in$ R-mod?

Toward this end, Chapter 2 develops some tools which will be used in later chapters. These tools involve a generalization of bounded Abelian groups due to Walker and Walker [20] and some results concerning torsion preradicals. Chapter 3 investigates the lattice of torsion classes. Chapter 4 leads to a theorem providing necessary and sufficient conditions on a certain subclass of torsion cyclic R-modules for a decomposition to occur. Separating examples are given to show that the theorems are meaningful. Finally, Chapter 5 studies completely reducible torsion classes and eventually yields two additional characterizations of rings studied by Dlab [8].

All homological concepts and basic facts concerning them may be found in any standard text; e.g. [14].

CHAPTER 2

Preliminaries

Every left ideal of R is an element of the power set of R. Then every topologizing and idempotent filter of left ideals of R is a subset of the power set of R, and is an element of the power set of the power set of R. The collection of all such filters is a subset of the power set of the power set of R, and is consequently a set. Since the collection of torsion classes is equipotent to this set, it must be a set also.

Chapter 3 will determine a lattice structure on the set of torsion classes on R-mod for a fixed ring, R. Before attempting this, several indispensable tools will be developed in this chapter. Section 1 is devoted to developing some properties of uniformly negligible modules, and Section 2 investigates some aspects of torsion preradicals and the torsion classes they generate.

Section 1. Uniformly Negligible Modules.

Following Teply [19], make the following definition:

Definition 2.1. Let I be a left ideal of R, and let M be an R-module. M is called I-uniformly negligible (I-bounded) if, for all $m \in M$, there exists $\{r_1, \ldots, r_{n(m)}\} \subseteq R$ such that $\bigcap_{i=1}^{n(m)} (I:r_i)m = 0$ $(Im = 0)$. If $\mathfrak{J}$ is a torsion class, then an

R-module, M, is called *$\mathfrak{J}$-uniformly negligible* (*$\mathfrak{J}$-bounded*) if there is some $I \in F(\mathfrak{J})$ such that M is I-uniformly negligible (I-bounded).

The prefixes, I and $\mathfrak{J}$, in the above definition will be omitted whenever no confusion is possible. The following proposition was mentioned without proof in [19].

Proposition 2.2. If I is a left ideal of R, then the class of I-uniformly negligible R-modules is closed under submodules, homomorphic images, and arbitrary direct sums.

Proof. Let I be a left ideal of R, and let M be an I-uniformly negligible module. Assume that $N \subset M$, and let $x \in N$. Then $x \in M$, and there exists $\{r_1,\ldots,r_n\} \subseteq R$ such that $\bigcap_{i=1}^{n} (I:r_i)x = 0$. Thus, N is I-uniformly negligible. Moreover, if $x \in M$, then there exists $\{r_1,\ldots,r_m\} \subseteq R$ such that $\bigcap_{i=1}^{m} (I:r_i)x = 0$. Then $\bigcap_{i=1}^{m} (I:r_i)x \subseteq N$. Since x was arbitrary, M/N is I-uniformly negligible.

Now let $\{M_\alpha\}_{\alpha\in\mathfrak{a}}$ be an arbitrary collection of I-uniformly negligible modules. Let $M = \bigoplus_{\alpha\in\mathfrak{a}} M_\alpha$, and let $x \in M$. Then $x = m_1 + m_2 + \ldots + m_k$, where $m_i \in M_{\alpha_i}$. For each m_i, there exists $\{r_1,\ldots,r_{n_i}\} = A_i \subseteq R$ such that $\bigcap_{r\in A_i} (I:r)m_i = 0$. If $B = \bigcup_{i=1}^{k} A_i$, then $\bigcap_{r\in B} (I:r)x = 0$, and M is I-uniformly negligible. $\square$

Proposition 2.3. If I is a two-sided ideal, then a module, M, is I-bounded if and only if M is I-uniformly negligible.

Proof. The proof follows immediately from the fact that $I \subseteq (I:r)$ whenever I is a two-sided ideal and $r \in R$. □

It follows that, for any torsion class over a left duo ring, the concepts of uniformly negligible and bounded are equivalent. Also notice that both of the above concepts generalize the concept of a bounded Abelian group.

Definition 2.4 [13]. A torsion class, $\mathfrak{J}$, is called a TTF class if it is closed under arbitrary direct products. In this case, the torsion theory with $\mathfrak{J}$ as its torsion class is also called TTF.

Proposition 2.5 [13, Theorem 2.1]. A torsion class, $\mathfrak{J}$, is TTF if and only if there exists a minimal ideal in $F(\mathfrak{J})$. In this event the minimal left ideal in $F(\mathfrak{J})$ is an idempotent, two-sided ideal of R.

Proposition 2.6. Let $\mathfrak{J}$ be a torsion class. Then $\mathfrak{J}$ is a TTF class if and only if all torsion modules are bounded.

Proof. The proof is immediate and will be omitted.

Example. The following is an example of a torsion class which is not TTF, but in which all torsion modules are uniformly negligible. The ring was studied by Cozzens [4].

Let $\overline{F}_2$ denote an algebraic closure of $Z/(2)$, and let ρ be the automorphism of $\overline{F}_2$ defined by $\rho(z) = z^2$. Let $\overline{F}_2[t,\rho]$ denote the ring of "twisted" polynomials in t over $\overline{F}_2$. The elements in $\overline{F}_2[t,\rho]$ are precisely the elements of $\overline{F}_2[t]$.

The addition is the same as in the polynomial ring, and the multiplication is determined by the identity $ta = \rho(a)t$ for all $a \in \overline{F}_2$ together with the distributive laws. The ring, $\overline{F}_2[t,\rho]$, is a principal left and right ideal domain [4].

Let $M = \{t^k \mid k$ is a non-negative integer$\}$. Let $R_M = \{\frac{a}{m} \mid a \in \overline{F}_2[t,\rho]$ and $m \in M\}$. Make the identity $\frac{a}{t^k} = \frac{b}{t^{k+i}}$ if $b = t^i a$. Addition is defined by $a/t^k + b/t^k = \frac{a+b}{t^k}$, and multiplication is defined by $\frac{a}{t^i} \frac{b}{t^j} = \frac{\rho^j(a)b}{t^{i+j}}$, where $\rho^j(a)$ is the element of $\overline{F}_2[t,\rho]$ obtained by applying the j^{th} iterate of ρ to each coefficient of a. Cozzens [4] showed that R_M is a simple principal right and left ideal domain, and there is a unique simple R_M-module up to isomorphism, which is injective. Furthermore, every right ideal is of the form $\prod_{i=1}^{n} (t-\alpha_i)R_M$, where $\alpha_i \in \overline{F}_2[t,\rho]\backslash\{0\}$.

Lemma 2.7. If I is a non-zero right ideal of R_M, then R_M/I is isomorphic to a finite direct sum of simple modules.

Proof. If I is a non-zero right ideal of R_M, then $I = \prod_{i=1}^{n} (t-\alpha_i)R_M$ where $\alpha_i \neq 0$ for all i. The proof of the lemma will follow by induction on n.

If $n = 1$, then the assertion is true by Lemma 2.2 of [4]. Assume that the assertion is true for all right ideals of R_M of the form $\prod_{i=1}^{n} (t - \alpha_i)R_M$. Let $I = \prod_{i=1}^{n+1} (t - \alpha_i)R_M$. Recall that for any module, P, the socle of P, written $\operatorname{soc}(P)$, is the sum of the simple submodules of P. Consider the exact sequence

$$0 \to \frac{\prod_{i=1}^{n}(t-\alpha_i)R_M}{\prod_{i=1}^{n+1}(t-\alpha_i)R_M} \to \frac{R_M}{\prod_{i=1}^{n+1}(t-\alpha_i)R_M} \to \frac{R_M}{\prod_{i=1}^{n}(t-\alpha_i)R_M} \to 0.$$

The left side is $(t - \alpha_{n+1})R_M$ bounded and is therefore its own socle. Since every simple module is injective and R_M is Noetherian, the left side is injective. Hence the sequence splits. Since the right side is a direct sum of simple modules, the assertion is verified. □

Consider the simple torsion class, $\mathcal{S}$, on right R_M modules. By Lemma 2.7 every proper right ideal is in $F(\mathcal{S})$. Since R_M is not a division ring, $\mathrm{soc}(R_M) = 0$. Therefore, $0 \notin F(\mathcal{S})$. Then $\mathcal{S}$ cannot be a TTF class, because R_M is a simple ring and because the minimal right ideal in the filter for a TTF class is a two-sided ideal by Proposition 2.5.

Let N be any maximal right ideal of R_M. If T is any module in $\mathcal{S}$ and $x \in T$, then $x \in \mathrm{soc}T$. Therefore, $T = \mathrm{soc}T$. Since each simple module is isomorphic to R_M/N, and N-uniformly negligible module, Proposition 2.2 yields that T is an N-uniformly negligible module. □

The final lemma in this section gives the structure of M-uniformly negligible modules, where M is a maximal left ideal of R.

LEMMA 2.8. Let M be a maximal left ideal of R. Then every M-uniformly negligible module is equal to a direct sum of cyclic modules, each of which is isomorphic to R/M.

Proof. It is sufficient to prove that every cyclic M-uniformly negligible module is isomorphic to a direct sum of copies of R/M. Let $\{r_1,r_2,\ldots,r_n\} \subseteq R$. Claim that $R/\bigcap_{i=1}^{n}(M:r_i)$ is isomorphic to a direct sum of copies of R/M. The proof will follow by induction on n. If $n = 1$, then the assertion is immediate. Assume that the assertion is true for all subsets of R of cardinality less than n. Let $\{r_1,\ldots,r_n\} \subseteq R$. Without loss of generality, one may assume that none of the r_i is in M. Call the subset irredundant if $\bigcap_{i\neq j}(M:r_i) \not\subseteq (M:r_j)$ for $j = 1,2,\ldots,n$. Let $J = \{r_1,r_2,\ldots,r_n\}$ and let $K = \{r_2,r_3,\ldots,r_n\}$. If J is not irredundant, then the assertion is true by the induction hypothesis. Assume that J is irredundant. Then

$$\frac{R}{\bigcap_{i\in J}(M:r_i)} = \frac{(M:r_1)+\bigcap_{i\in K}(M:r_i)}{\bigcap_{i\in J}(M:r_i)} = \frac{(M:r_1)}{\bigcap_{i\in J}(M:r_i)} \oplus \frac{\bigcap_{i\in K}(M:r_i)}{\bigcap_{i\in J}(M:r_i)} \cong \frac{R}{\bigcap_{i\in K}(M:r_i)}$$

$$\oplus \frac{R}{(M:r_1)} \ .$$

However, $R/\bigcap_{i\in K}(M:r_i)$ satisfies the conditions of the induction hypothesis and is therefore isomorphic to a direct sum of copies of R/M. $R/(M:r_i)$ is isomorphic to R/M so that the induction is complete. The proof is completed by noting that every cyclic M-uniformly negligible module is a homomorphic image of a module of the above form. □

Section 2. Torsion Preradicals.

Definition 2.9 [10]. A subfunctor, ρ, of the identity functor from R-mod to R-mod is said to be a torsion preradical if and only if the following conditions are satisfied:

(1) If $\alpha: M \to N$ is a homomorphism, then $\alpha(\rho(M)) \subseteq \rho(N)$;

(2) If $N \subseteq M$, then $\rho(N) = N \cap \rho(M)$.

If $M = \rho(M)$, then M is called a ρ-module.

Proposition 2.10. A subfunctor, ρ, of the identity functor on R-mod is a torsion preradical if and only if the following conditions are satisfied:

(1) $\rho^2 = \rho$;

(2) $N \subseteq M$ implies that $\rho(N) \subseteq \rho(M)$;

(3) The class of ρ-modules is closed under both submodules and homomorphic images.

Proof. ($\Rightarrow$). Let M be a ρ-module, and let $N \subseteq M$. Then $\rho(N) = N \cap \rho(M) = N \cap M = N$. Hence N is also a ρ-module. Let π be the natural map from M onto M/N. Since $M/N = \pi(M) = \pi(\rho(M)) \subseteq \rho(M/N)$, M/N is a ρ-module. Therefore, the class of ρ-modules is closed under submodules and homomorphic images. In [10], Fuchs showed that $\rho^2 = \rho$. If $N \subseteq M$, then condition (2) of the definition implies that $\rho(N) \subseteq \rho(M)$.

Conversely, assume that ρ satisfies the three conditions of this proposition. Let M be an R-module, and let $N \subseteq M$. $\rho(N) \subseteq \rho(M) \cap N$. Since ρ is idempotent and the class of ρ-modules is closed under submodules, $\rho(M) \cap N = \rho(\rho(M) \cap N)$.

Consequently, $\rho(N) = \rho(M) \cap N$, because every ρ-module contained in M is also contained in $\rho(M)$. Condition (1) of the definition follows from the fact that the class of ρ-modules is closed under homomorphic images. $\square$

Let ρ be a torsion preradical. For any module, M, define $\rho_0(M) = 0$. Let α be an ordinal number, and assume that for all ordinal numbers, $\beta < \alpha$, $\rho_\beta(M)$ is defined. If $\rho_\beta(M)$ is defined, then let π_β be the natural map from M onto $M/\rho_\beta(M)$. Suppose that $\alpha = \beta + 1$. Define $\rho_\alpha(M) = (\pi_\beta)^{-1}(\rho(M/\rho_\beta(M)))$. If α is a limit ordinal, then define $\rho_\alpha(M) = \bigcup_{\beta<\alpha} \rho_\beta(M)$. For every module, M, there is a smallest ordinal, $\lambda(M)$, such $\rho_{\lambda(M)}(M) = \rho_{\lambda(M)+1}(M)$. Let $\mathcal{T} = \{M | \rho_{\lambda(M)}(M) = M\}$. The two following propositions concerning this construction appear in [10] without proof. The proofs will be included here for completeness.

Proposition 2.11. If ρ_α is defined as above, then ρ_α is a torsion preradical.

Proof. If $\alpha = 1$, then ρ_α is a torsion preradical by the definition of ρ_1. Assume that ρ_β is a torsion preradical for all $\beta < \alpha$. Assume that $\alpha = \beta + 1$. Let $f:M \to N$ be a homomorphism. Then $\alpha(\rho_\beta(M)) \subseteq \rho_\beta(N)$. Consider the induced homomorphism $\overline{f}:M/\rho_\beta(M) \to N/\rho_\beta(N)$. Then $\overline{f}(\rho(M/\rho_\beta(M)) \subseteq \rho(N/\rho_\beta(N))$. Therefore, $f(\rho_\alpha(M)) \subseteq \rho_\alpha(N)$. Assume that $N \subseteq M$. $\rho_\beta(N) = N \cap \rho_\beta(M)$. Let $x \in N \cap \rho_\alpha(M)$. Then $x + \rho_\beta(M) \in \rho(\frac{N + \rho_\beta(M)}{\rho_\beta(M)})$, which implies that $x + (N \cap \rho_\beta(M)) \in \rho(N/N \cap \rho_\beta(M)) = \rho(N/\rho_\beta(N))$;

therefore $x \in \rho_\alpha(N)$. Hence ρ_α is a torsion preradical. If α is a limit ordinal and $f:M \to N$, then $f(\rho_\alpha(M)) = f(\bigcup_{\beta<\alpha} \rho_\beta(M)) \subseteq \bigcup_{\beta<\alpha} \rho_\beta(N) = \rho_\alpha(N)$. If $x \in N \cap \rho_\alpha(M)$, then $x \in N \cap \rho_\beta(M)$ for some $\beta < \alpha$. Thus $x \in \rho_\beta(N) \subseteq \rho_\alpha(N)$. □

Proposition 2.12. Let ρ be a torsion preradical, and let ρ_α be defined as above, where α is an ordinal number. Define $\mathfrak{T}(M) = \rho_{\lambda(M)}(M)$. If $\mathfrak{T} = \{M \mid \rho_{\lambda(M)}(M) = M\}$, then $\mathfrak{T}$ is the smallest torsion class containing the class of ρ-modules.

Proof. The first objective is to show that $\mathfrak{T}$ is a torsion class. Let $M \in \mathfrak{T}$, and let $N \subseteq M$. Since $\rho_{\lambda(M)}$ is a torsion preradical, $\rho_{\lambda(M)}(N) = \rho_{\lambda(M)}(M) \cap N = M \cap N = N$. Since M is a $\rho_{\lambda(M)}$-module, Proposition 2.10 implies that M/N is a $\rho_{\lambda(M)}$-module also. Therefore $\mathfrak{T}$ is closed under submodules and homomorphic images. Let $\{M_\alpha\}_{\alpha\in\mathfrak{A}}$ be a collection of $\mathfrak{T}$-modules. For each $\alpha \in \mathfrak{A}$, $\text{card}(M_\alpha) \geq \lambda(M_\alpha)$. Therefore, $\lambda(M_\alpha) \leq \text{card}(\bigoplus_{\alpha\in\mathfrak{A}} M_\alpha) = \gamma$ for all $\alpha \in \mathfrak{A}$. By [10, Lemma 1], $\rho_\gamma(\bigoplus_{\alpha\in\mathfrak{A}} M_\alpha) = \bigoplus_{\alpha\in\mathfrak{A}} \rho_\gamma(M_\alpha) = \bigoplus_{\alpha\in\mathfrak{A}} M_\alpha$. Therefore, $\mathfrak{T}(\bigoplus_{\alpha\in\mathfrak{A}} M_\alpha) = \bigoplus_{\alpha\in\mathfrak{A}} M_\alpha$, and $\mathfrak{T}$ is closed under arbitrary direct sums.

Finally, it is necessary to show that $\mathfrak{T}$ is closed under extensions. Toward this goal, let

$$0 \to N \to M \to M/N \to 0$$

be an exact sequence such that N and M/N are in $\mathfrak{T}$. If α is the smallest ordinal such that $m + N \in \rho_\alpha(M/N)$, then $m \in \rho_{\lambda(N)+\alpha}(M)$. The proof of this assertion will follow by transfinite induction on α. If $\alpha = 1$, then

$$m \in \pi_1^{-1}(\rho(M/N)) \subseteq \pi_0^{-1}(\rho(\frac{M}{\rho_{\lambda(N)}(M)}) = \rho_{\lambda(N)+1}(M),$$

where π_1 and π_0 are the appropriate natural maps. Assume that the assertion is true for all ordinals $\beta < \alpha$ and all $N \subseteq M$. Let $\alpha = \beta + 1$ be the smallest ordinal such that $m + N \in \rho_\alpha(M/N) = (\pi^{-1})(\rho(\frac{M/N}{\rho_\beta(M/N)}))$, where π is the natural map from M/N onto $\frac{M/N}{\rho_\beta(M/N)}$. Let $\rho_\beta(M/N) = L/N$. By the induction hypothesis, $L \subseteq \rho_{\lambda(N)+\beta}(M)$. But

$$m + N \in (\pi_0^{-1})(\rho(M/L)) \subseteq (\pi_1^{-1})\rho(\frac{M}{\rho_{\lambda(N)+\beta}(M)}) = \rho_{\lambda(N)+\beta+1}(M) = \rho_{\lambda(N)+\alpha}(M),$$

where π_0 and π_1 are the natural maps from M/N onto M/L and $M/\rho_{\lambda(N)+\beta}(M)$ respectively. Since α cannot be a limit ordinal, the induction is complete. This completes the original assertion, and $\mathfrak{T}$ is a torsion class.

Let $\mathfrak{T}'$ be a torsion class which contains the class of ρ-modules. If $M \in \mathfrak{T}$, then $M \in \mathfrak{T}'$. The proof follows by a routine transfinite induction on $\lambda(M)$ and will be omitted. □

CHAPTER 3

The Lattice of Torsion Classes

Let R be a fixed ring with identity. Consider the set, $\mathcal{L}$, of torsion classes, and order $\mathcal{L}$ by inclusion. Define two binary operations, $\vee$ and $\wedge$, and $\mathcal{L}$ as follows: $\mathcal{T}_1 \vee \mathcal{T}_2$ is the smallest torsion class containing the set theoretic union, $\mathcal{T}_1 \cup \mathcal{T}_2$, and $\mathcal{T}_1 \wedge \mathcal{T}_2$ is the largest torsion class contained in the intersection $\mathcal{T}_1 \cap \mathcal{T}_2$. Under these operations $\mathcal{L}$ becomes a lattice, which will be denoted by $\{\mathcal{S},\vee,\wedge\}$. $\{\mathcal{L},\vee,\wedge\}$ has a zero (namely $\{0\}$) and a 1 (namely R-mod). It is necessary to examine the meet and join operations a little more closely.

The meet operation is more accessible and consequently will be treated first. The following proposition yields the result that the meet of two torsion classes is precisely the set theoretic intersection of the classes. In fact, the result shows that any collection of torsion classes has a greatest lower bound and consequently a meet.

Proposition 3.1. If $\{\mathcal{T}_\alpha\}_{\alpha\in\mathcal{A}}$ is any non-empty collection of torsion classes, then $\bigcap_{\alpha\in\mathcal{A}} \mathcal{T}_\alpha$ is also a torsion class.

Proof. It is sufficient to verify that the class, $\bigcap_{\alpha\in\mathcal{A}} \mathcal{T}_\alpha$, is closed under submodules, homomorphic images, arbitrary direct sums, and extensions. Since each $\mathcal{T}_\alpha$ is closed under these properties, the proof is routine and will be omitted.$\square$

The join of two torsion classes may be constructed in two ways, internally and externally. The internal construction is somewhat more complicated than the external one, but it has the advantage of being much more instructive. Since it is simpler, the external construction of the join will be made first.

Proposition 3.2. The join of two torsion classes, $\mathfrak{T}_1$ and $\mathfrak{T}_2$, is the intersection of all torsion classes which contain $\mathfrak{T}_1 \cup \mathfrak{T}_2$.

Proof. R-mod is a torsion class which contains $\mathfrak{T}_1 \cap \mathfrak{T}_2$. Hence $\mathfrak{C} = \{\mathfrak{T} \in \{\mathcal{L},\vee,\wedge\} \mid \mathfrak{T} \supseteq \mathfrak{T}_1 \cup \mathfrak{T}_2\}$ is a non-empty set. By Proposition 3.1, $\mathfrak{U} = \bigcap_{\mathfrak{T}\in\mathfrak{C}} \mathfrak{T}$ is a torsion class. Clearly $\mathfrak{T}_1 \cup \mathfrak{T}_2 \subseteq \mathfrak{U}$. If $\mathfrak{V}$ is any torsion class containing $\mathfrak{T}_1 \cup \mathfrak{T}_2$, then $\mathfrak{V} \in \mathfrak{C}$. Hence $\mathfrak{U} \subseteq \mathfrak{V}$. Hence $\mathfrak{U}$ is the smallest torsion class containing $\mathfrak{T}_1 \cup \mathfrak{T}_2$, which means $\mathfrak{U}$ is the join of $\mathfrak{T}_1$ and $\mathfrak{T}_2$. □

Before the internal construction of $\mathfrak{T}_1 \vee \mathfrak{T}_2$ can be made, a preliminary result is required.

Lemma 3.3. Let $\mathfrak{T}$ and $\mathfrak{T}_1$ be torsion classes. For all $M \in$ R-mod, define $\rho(M) = (\pi_M^{-1})(\mathfrak{T}(M/\mathfrak{T}_1(M)))$, where π_M is the natural map from M onto $M/\mathfrak{T}_1(M)$. Then ρ is a torsion preradical.

Proof. ρ is a subfunctor of the identity functor, because $\mathfrak{T}$ and $\mathfrak{T}_1$ are torsion classes. Let $M = \rho(M)$. Then

$$\rho(\rho(M)) = (\pi_M^{-1})(\mathfrak{T}[\rho(M)/\mathfrak{T}_1(\rho(M))]) = \pi_M^{-1}(\mathfrak{T}[M/\mathfrak{T}_1(M)]) = \rho(M).$$

Hence ρ is idempotent. Let $N \subseteq M$. If $x \in \rho(N)$, then $x + \mathfrak{T}_1(N) \in \mathfrak{T}(N/\mathfrak{T}_1(N))$. Therefore, $(\mathfrak{T}_1(N):x) \in F(\mathfrak{T})$. However, $(\mathfrak{T}_1(M):x) = (\mathfrak{T}_1(N):x)$, because $\mathfrak{T}_1$ is a torsion preradical. Thus, $x + \mathfrak{T}_1(M) \in \mathfrak{T}(M/\mathfrak{T}_1(M))$, which implies that $x \in \rho(M)$. Since this is true for all $x \in \rho(N)$, $\rho(N) \subseteq \rho(M)$.

By Proposition 2.10, it suffices to show that the class of ρ-modules is closed under both submodules and homomorphic images. Let $M = \rho(M)$, and let $N \subseteq M$. If $x \in N$, then $(\mathfrak{T}_1(N):x) = (\mathfrak{T}_1(M):x) \in F(\mathfrak{T})$. Thus, $x \in \rho(N)$, and N is a ρ-module. Furthermore, if $x \in M$, then $(\mathfrak{T}_1(M):x) \in F(\mathfrak{T})$. Since $(\mathfrak{T}_1(M) + N)/N \subseteq \mathfrak{T}_1(M/N)$, $x \in \rho(M/N)$. □

By constructing the sequence of torsion preradicals, ρ_α, one obtains a torsion class, $\mathfrak{U}$, which contains the class of ρ-modules. In fact, by Proposition 2.10, $\mathfrak{U}$ is the smallest torsion class containing the class of ρ-modules.

Proposition 3.4. If $\mathfrak{U}$ is defined as above, then $\mathfrak{U} = \mathfrak{T} \vee \mathfrak{T}_1$.

Proof. Since all modules in $\mathfrak{T} \cup \mathfrak{T}_1$ are ρ-modules, $\mathfrak{T} \vee \mathfrak{T}_1 \subseteq \mathfrak{U}$. Let $\mathfrak{V}$ be any torsion class containing $\mathfrak{T} \cup \mathfrak{T}_1$, and let $M = \rho(M)$. Consider the exact sequence

$$0 \to \mathfrak{T}_1(M) \to M \to M/\mathfrak{T}_1(M) \to 0.$$

By hypothesis $M/\mathfrak{T}_1(M) \in \mathfrak{T}$. Consequently, since M is the extension of two $\mathfrak{V}$-modules, $M \in \mathfrak{V}$. By the remark preceding this proposition, $\mathfrak{U} \subseteq \mathfrak{V}$. Since $\mathfrak{T} \vee \mathfrak{T}_1$ contains $\mathfrak{T} \cup \mathfrak{T}_1$, the proposition is completed. □

An interesting aspect of this construction is that, in some sense, it is not commutative. That is, if $\mathfrak{T}$ and $\mathfrak{T}_1$ are torsion classes, and for any module, M, π_M is the natural map from M onto $M/\mathfrak{T}(M)$, and ρ_M is the natural map from M onto $M/\mathfrak{T}_1(M)$, then $(\pi_M^{-1})(\mathfrak{T}[M/\mathfrak{T}_1(M)])$ does not necessarily equal $(\rho_M^{-1})(\mathfrak{T}_1[M/\mathfrak{T}(M)])$. However, after the transfinite construction, the torsion classes turn out to be the same. An example of the noncommutativity occurs when $R = \prod_{i=1}^{\infty} F_i$, where each F_i is an isomorphic copy of some fixed field, F. Let $\mathcal{S}$ be the simple torsion class, and let $\mathcal{G}$ be the Goldie torsion class. $(\pi_R^{-1})(\mathcal{S}[R/\mathcal{G}(R)]) = \bigoplus_{i=1}^{\infty} F_i$ and $(\rho_R^{-1})(\mathcal{G}[R/\mathcal{S}(R)]) = R$. It is essential for studying decompositions of torsion classes to decide when the construction in commutative. In this regard, consider the following two results.

Lemma 3.5. Let $\mathfrak{T}$ and $\mathfrak{T}_1$ be torsion classes on R-mod. Then the following statements are equivalent:

(1) $\mathfrak{T} \cap \mathfrak{T}_1 = \{0\}$.

(2) If $I \in F(\mathfrak{T})$ and $J \in F(\mathfrak{T}_1)$, then $I + J = R$.

(3) There are no non-zero simple modules in $\mathfrak{T} \cap \mathfrak{T}_1$.

Proof. (1) ⇒ (2). Let $I \in F(\mathfrak{T})$, and let $J \in F(\mathfrak{T}_1)$. Then $R/I+J$ is a homomorphic image of both R/I and R/J. Thus, $R/I+J \in \mathfrak{T} \cap \mathfrak{T}_1 = \{0\}$. Consequently, $R = I + J$.

(2) ⇒ (3). Let S be a non-zero simple module in $\mathfrak{T} \cap \mathfrak{T}_1$. $S \cong R/M$, for some maximal left ideal, M. Then $M \in F(\mathfrak{T}) \cap F(\mathfrak{T}_1)$. By condition (2), $R = M + M = M$, which contradicts M being proper.

(3) ⇒ (1). Let $M \in \mathfrak{J} \cap \mathfrak{J}_1$. If M contains a non-zero cyclic submodule, Rm, then $Rm \in \mathfrak{J} \cap \mathfrak{J}_1$ also. $Rm \cong R/L$, where L is a proper left ideal of R. By a Zorn's Lemma argument, L is contained is some maximal left ideal N. R/N is a simple homomorphic image of R/L. Thus, $R/N \in \mathfrak{J} \cap \mathfrak{J}_1$, which contradicts condition (3). □

Proposition 3.6. Let $\mathfrak{J}$ and $\mathfrak{J}_1$ be torsion classes such that $\mathfrak{J} \cap \mathfrak{J}_1 = \{0\}$. For all modules, M, let π_M be the natural map from M onto $M/\mathfrak{J}(M)$, and let ρ_M be the natural map from M onto $M/\mathfrak{J}_1(M)$. Then the following statements are equivalent:

(1) $(\rho_M^{-1})(\mathfrak{J}[M/\mathfrak{J}_1(M)]) = (\pi_M^{-1})(\mathfrak{J}_1[M/\mathfrak{J}(M)])$, for all $M \in R\text{-mod}$.

(2) $(\rho_M^{-1})(\mathfrak{J}[M/\mathfrak{J}_1(M)]) = \mathfrak{J}(M) \oplus \mathfrak{J}_1(M) = (\pi_M^{-1})(\mathfrak{J}_1[M/\mathfrak{J}(M)])$, for all $M \in R\text{-mod}$.

(3) $\mathfrak{J}(M/[\mathfrak{J}(M) \oplus \mathfrak{J}_1(M)]) = \{0\} = \mathfrak{J}_1(M/[\mathfrak{J}(M) \oplus \mathfrak{J}_1(M)])$, for all $M \in R\text{-mod}$.

(4) $(\mathfrak{J} \vee \mathfrak{J}_1)(M) = \mathfrak{J}(M) \oplus \mathfrak{J}_1(M)$, for all $M \in R\text{-mod}$.

Proof. (1) ⇒ (2). If $m \in (\rho_M^{-1})(\mathfrak{J}[M/\mathfrak{J}_1(M)])$, where M is an R-module, then there exists $I \in F(\mathfrak{J})$ such that $Im \subseteq \mathfrak{J}_1(M)$. By condition (1), $m \in (\pi_M^{-1})(\mathfrak{J}_1[M/\mathfrak{J}(M)])$. There exists $J \in F(\mathfrak{J}_1)$ such that $Jm \subseteq \mathfrak{J}(M)$. Since $\mathfrak{J} \cap \mathfrak{J}_1 = \{0\}$, two facts are immediate One fact is that $\mathfrak{J}(M) + \mathfrak{J}_1(M)$ is always a direct sum. The second fact, which follows from Lemma 3.5, is that $I + J = R$. Thus, $Rm = (I + J)m \subseteq Im + Jm \subseteq \mathfrak{J}_1(M) \oplus \mathfrak{J}(M)$. Since m was arbitrary, $(\rho_M^{-1})(\mathfrak{J}[M/\mathfrak{J}_1(M)]) \subseteq \mathfrak{J}(M) \oplus \mathfrak{J}_1(M)$.

Conversely, let $m = m_1 + m_2 \in \mathcal{T}(M) \oplus \mathcal{T}_1(M)$. Then $(0:m_1)m \subseteq \mathcal{T}_1(M)$. By the way m_1 was chosen $(0:m_1) \in F(\mathcal{T})$. Therefore, $m + \mathcal{T}_1(M) \in \mathcal{T}(M/\mathcal{T}_1(M))$. The other equality is proved in an entirely analagous manner.

(2) ⇒ (3). Let M be an R-module. For all m such that $m + (\mathcal{T}(M) \oplus \mathcal{T}_1(M)) \in \mathcal{T}(M/[\mathcal{T}(M) \oplus \mathcal{T}_1(M)])$, consider the exact sequence

$$0 \to \frac{\mathcal{T}(M) \oplus \mathcal{T}_1(M)}{\mathcal{T}_1(M)} \to \frac{Rm + (\mathcal{T}(M) \oplus \mathcal{T}_1(M))}{\mathcal{T}_1(M)} \to \frac{Rm + (\mathcal{T}(M) \oplus \mathcal{T}_1(M))}{\mathcal{T}(M) \oplus \mathcal{T}_1(M)} \to 0.$$

Since both ends of the sequence are in $\mathcal{T}$, the middle must be a member of $\mathcal{T}$ also. By condition (2), $m \in \mathcal{T}(M) \oplus \mathcal{T}_1(M)$. The other equality may be obtained in a similar manner.

(3) ⇒ (1). Let M be an R-module. If $m + \mathcal{T}_1(M) \in \mathcal{T}(M/\mathcal{T}_1(M))$, then $m + (\mathcal{T}(M) \oplus \mathcal{T}_1(M)) \in \mathcal{T}(M/[\mathcal{T}(M) \oplus \mathcal{T}_1(M)])$. By condition (3), $m \in \mathcal{T}(M) \oplus \mathcal{T}_1(M)$. Consequently, $(\rho_M^{-1})(\mathcal{T}[M/\mathcal{T}_1(M)]) \subseteq \mathcal{T}(M) \oplus \mathcal{T}_1(M)$. The reverse inclusion follows from the proof of (1) ⇒ (2). An entirely analagous argument yields that $(\pi_M^{-1})(\mathcal{T}_1[M/\mathcal{T}(M)]) = \mathcal{T}(M) \oplus \mathcal{T}_1(M)$.

(3) ⇒ (4). In this case conditions (1) and (2) may also be assumed. Let $\mathcal{P}(M) = (\rho_M^{-1})(\mathcal{T}[M/\mathcal{T}_1(M)])$ for all $M \in$ R-mod. By Lemma 3.3 and Proposition 3.4, $\mathcal{P}$ is a torsion preradical which generates $\mathcal{T} \vee \mathcal{T}_1$. By condition (2), $\mathcal{P}(M) = \mathcal{T}(M) \oplus \mathcal{T}_1(M)$. However,

$$\mathcal{P}_2(M) = (\sigma_M^{-1})(\mathcal{P}[M/\mathcal{P}(M)]) = (\sigma_M^{-1})(0) = \mathcal{T}(M) \oplus \mathcal{T}_1(M),$$

where σ_M is the natural map from M onto $M/\mathcal{P}(M)$. Since $\mathcal{P}_2(M) = \mathcal{P}(M)$ for all $M \in$ R-mod, $\mathcal{T}(M) \oplus \mathcal{T}_1(M) = (\mathcal{T} \vee \mathcal{T}_1)(M)$.

(4) ⇒ (1). This is immediate from the construction of $\mathfrak{J} \vee \mathfrak{J}_1$. □

It was noted earlier that any arbitrary collection of torsion classes has a greatest lower bound; namely their intersection. Any collection of torsion classes also has a least upper bound; namely the intersection of all torsion classes containing their set theoretic union. Hence the lattice of torsion classes is a complete lattice. Once again, this construction is not very instructive, and an internal construction will be given. Toward this end we have the following result.

Proposition 3.7. Let $\{\mathfrak{J}_\alpha\}_{\alpha\in\mathfrak{a}}$ be a collection of torsion classes such that any two elements of the collection are comparable. Define $\rho(M) = \sum_{\alpha\in\mathfrak{a}} \mathfrak{J}_\alpha(M)$ for all $M \in$ R-mod. Then ρ is a torsion preradical. If, in addition, for all $I \in F(\mathfrak{J}_\alpha)$, there exists a finitely generated $J \in F(\mathfrak{J}_\beta)$ such that $J \subseteq I$, then the class of ρ-modules is a torsion class which is the join of $\{\mathfrak{J}_\alpha\}_{\alpha\in\mathfrak{a}}$.

Proof. Since any two elements of $\{\mathfrak{J}_\alpha\}_{\alpha\in\mathfrak{a}}$ are comparable, ρ is a subfunctor of the identity functor on R-mod. Let M be an R-module. Then

$$\rho(\rho(M)) = \sum_{\alpha\in\mathfrak{a}} \mathfrak{J}_\alpha(\rho(M)) = \sum_{\alpha\in\mathfrak{a}} (\mathfrak{J}_\alpha(M) \cap \rho(M)) = \sum_{\alpha\in\mathfrak{a}} \mathfrak{J}_\alpha(M) = \rho(M).$$

If $N \subseteq M$, then

$$\rho(N) = \sum_{\alpha\in\mathfrak{a}} \mathfrak{J}_\alpha(N) = \sum_{\alpha\in\mathfrak{a}} \mathfrak{J}_\alpha(M) \cap N \subseteq \sum_{\alpha\in\mathfrak{a}} \mathfrak{J}_\alpha(M) = \rho(M).$$

By Proposition 2.10, it is sufficient to show that the collection of ρ-modules is closed under submodules and homomorphic images.

Suppose that $M = \rho(M)$, and that $N \subseteq M$. Let $m \in N$. From the definition of ρ, $m = m_1 + \ldots\ m_k$, where each $m_i \in \mathcal{T}_{\alpha_i}$. Since any two of these torsion classes are comparable, $m \in \mathcal{T}_{\alpha_j}$ for some j. Consequently, $m \in \mathcal{T}_{\alpha_j}(M) \cap N = \mathcal{T}_{\alpha_j}(N) \subseteq \rho(N)$. If $m \in M$, then $m \in \mathcal{T}_{\alpha_o}(M)$ for some $\alpha_o \in \mathcal{C}$. Thus,

$$m + N \in \mathcal{T}_{\alpha_o}(M/N) \subseteq \sum_{\alpha \in \mathcal{C}} \mathcal{T}_{\alpha}(M/N) = \rho(M/N).$$

For the second part of the proposition it suffices to show that the class of ρ-modules is closed under extensions because all torsion preradicals preserve direct sums [10]. Assume that

$$0 \to A \to B \to B/A \to 0$$

is an exact sequence with the ends being ρ-modules. If $b \in B$, then $b + A \in \mathcal{T}_{\alpha}(B/A)$ for some $\alpha \in \mathcal{C}$. By hypothesis, there exists a finitely generated left ideal, $J \in F(\mathcal{T}_{\beta})$, such that $Jb \subseteq A$. Assume that $J = \langle r_1, \ldots, r_n \rangle$. Then each $r_i b \in \mathcal{T}_{\alpha_i}(A)$ for some $\alpha_i \in \mathcal{C}$. Let $\mathcal{T}_{\gamma}$ be the maximum torsion class of $\{\mathcal{T}_{\alpha_1}, \ldots, \mathcal{T}_{\alpha_n}, \mathcal{T}_{\beta}\}$. Then $Jb \subseteq \mathcal{T}_{\gamma}(A)$. Consider the exact sequence

$$0 \to Jb \to Rb \to \frac{Rb}{Jb} \to 0.$$

The ends are in $\mathcal{T}_{\gamma}$, which implies that $Rb \in \rho(B)$. Since B was arbitrary, $B = \rho(B)$, and the proposition is complete. □

Notice that even if the extra hypothesis does not hold, one could construct the join of any collection of comparable torsion classes by first forming the torsion preradical, ρ, of Proposition 3.7, and by then observing that the join of the collection is the torsion class generated by ρ. What happens if the collection, $\{\mathfrak{T}_\alpha\}_{\alpha\in\mathfrak{A}}$, is not comparable? First, well-order $\mathfrak{A}$, so that $\mathfrak{A}$ might be thought of as a cardinal number. Re-index each $\mathfrak{T}_\alpha$ by letting $\mathfrak{T}_\alpha' = \mathfrak{T}_{\alpha+1}$. Let $\mathfrak{T}_\alpha = 0$ if α is a limit ordinal less than $\mathfrak{A}$. The new indices are certainly contained in $\mathfrak{A} + 1$, so that they, too, form a set. Call this set $\mathfrak{B}$. $\bigvee_{\alpha\in\mathfrak{A}} \mathfrak{T}_\alpha = \bigvee_{\beta\in\mathfrak{B}} \mathfrak{T}_\beta'$. Define $\mathfrak{U}_1 = \{0\}$, and let $\mathfrak{U}_2 = \mathfrak{T}_2'$. Assume that $\mathfrak{U}_\beta$ has been defined for all $\beta < \alpha$. If $\alpha = \beta + 1$, then define $\mathfrak{U}_\alpha = \mathfrak{U}_\beta \vee \mathfrak{T}_\alpha'$ if $\mathfrak{T}_\alpha'$ exists. If $\mathfrak{T}_\alpha'$ does not exist, then let $\mathfrak{U}_\alpha = \mathfrak{U}_\beta$. If α is a limit ordinal, then let $\mathfrak{U}_\alpha$ be the torsion class generated by the torsion preradical, $\sum_{\beta<\alpha} \mathfrak{U}_\beta(M)$. This is a torsion preradical, because the $\{\mathfrak{U}_\beta\}_{\beta<\alpha}$ is a comparable collection of torsion classes. Then $\{\mathfrak{U}_\beta\}_{\beta\in\mathfrak{B}}$ form a comparable class, so that $\rho(M) = \sum_{\beta\in\mathfrak{B}} \mathfrak{U}_\beta(M)$ is a torsion preradical which generates $\bigvee_{\beta\in\mathfrak{B}} \mathfrak{U}_\beta$. It is clear that $\bigvee_{\alpha\in\mathfrak{A}} \mathfrak{T}_\alpha = \bigvee_{\beta\in\mathfrak{B}} \mathfrak{U}_\beta$.

Since every torsion class is determined by the cyclic modules which are in it, each torsion class, $\mathfrak{T}$, is $\bigvee_{I\in F(\mathfrak{T})} \mathfrak{T}_{R/I}$, where *$\mathfrak{T}_{R/I}$ is the smallest torsion class containing R/I*. It would be interesting to construct $\mathfrak{T}_{R/I}$ for an arbitrary left ideal I, for this would lead to a way of characterizing torsion classes.

Lemma 3.8. Let I be a left ideal of R. For any R-module, M, define $\rho^I(M) = \{m \in M \mid$ There exist $\{r_1,\dots,r_n\} \subseteq R$ such that $\bigcap_{i=1}^{n} (I:r_i)m = 0\}$. Then ρ^I is a torsion preradical, and if ρ

is a torsion preradical such that $\rho(R/I) = R/I$, then $\rho^I(M) \subseteq \rho(M)$ for all $M \in R$-mod.

Proof. Let M be an R-module, and let m_1 and m_2 be elements of $\rho^I(M)$. Then there exist $\{r_1,\dots,r_n\} \subseteq R$ and $\{s_1,\dots,s_n\} \subseteq R$ such that $\bigcap_{i=1}^{n} (I:r_i)m_1 = 0$ and $\bigcap_{i=1}^{m} (I:s_i)m_2 = 0$, so that $[(\bigcap_{i=1}^{n} (I:r_i)) \cap (\bigcap_{i=1}^{m} (I:s_i))][m_1 + m_2] = 0$. If $r \in R$, then $\bigcap_{i=1}^{n} (I:rr_i)m_1 = 0$. Hence $\rho^I(M)$ is a submodule of M, for all $M \in R$-mod. Proposition 2.2 yields that ρ^I-modules are closed under submodules and homomorphic images. That ρ^I is idempotent follows immediately. Finally, assume that $N \subseteq M$, and that $x \in \rho^I(N)$. It is clear that $x \in \rho^I(M)$. By Proposition 2.10, ρ^I is a torsion preradical.

Let ρ be a torsion preradical such that $\rho(R/I) = R/I$. It is sufficient to show that every ρ^I-module is also a ρ-module. If $r \in R$, then the fact that $\frac{Rr + I}{I} \subseteq R/I$ implies that $R/(I:r) \cong \frac{Rr + I}{I}$ is a ρ-module. If $\{r_1,\dots,r_n\} \subseteq R$, then $\bigoplus_{i=1}^{n} R/(I:r_i)$ is a ρ-module. Map $R/\bigcap_{i=1}^{n} (I:r_i)$ into $\bigoplus_{i=1}^{n} (R/(I:r_i)$ by

$$1 + \bigcap_{i=1}^{n} (I:r_i) \longmapsto (1 + (I:r_1),\ 1 + (I:r_2),\dots,1 + (I:r_n)).$$

Since this map is a monomomorphism, $R/\bigcap_{i=1}^{n} (I:r_i)$ is a ρ-module. Since every ρ^I-module is a homomorphic image of a direct sum of modules of this form, every ρ^I-module is a ρ-module. □

Definition 3.9. If $\mathfrak{T}$ is a torsion class, then a collection, P, of left ideals of R is called a subbasis for $\mathfrak{T}$ if $\mathfrak{T} = \bigvee_{I \in P} \mathfrak{T}_{R/I}$. A subbasis, P, of $\mathfrak{T}$ is called a separating subbasis if $\mathfrak{T}_{R/I} \cap \bigvee_{J \in P\setminus\{I\}} \mathfrak{T}_{R/J} = \{0\}$ for all $I \in P$.

Every torsion class, $\mathfrak{T}$, has a subbasis, namely $F(\mathfrak{T})$. However, a torsion class may have a much smaller subbasis. For example, the simple torsion class has a subbasis consisting of any collection, P, of maximal left ideals such that every simple R-module is isomorphic to R/M for some $M \in P$. There are torsion classes which do not have a separating subbasis as the following example shows.

Example. Let $R = F[x_1,\ldots,x_n,\ldots]$ be the polynomial ring in countably many commuting indeterminants over a field, F. Localize R by the maximal ideal $M = (x_1,x_2,\ldots,x_n,\ldots)$, and call the localization R_M. Recall that $R_M = \{\frac{a}{b} \mid a \in R$ and $b \in R\setminus M\}$ with multiplication and addition defined in the obvious manner. R_M is a local ring with unique maximal $I_1 = (\frac{x_1}{1}, \frac{x_2}{1}, \ldots, \frac{x_n}{1}, \ldots)$. Let $I_n = (\frac{x_n}{1}, \frac{x_{n+1}}{1}, \frac{x_{n+2}}{1}, \ldots)$ for all $n = 1, 2, \ldots$. The collection of I_n forms a decreasing chain $I_1 \supsetneq I_2 \supsetneq \cdots \supsetneq I_n \supsetneq \cdots$. Consequently, $\mathfrak{T}_{R/I_1} \subseteq \mathfrak{T}_{R/I_2} \subseteq \cdots \subseteq \mathfrak{T}_{R/I_n} \subseteq \cdots$. Let $\mathfrak{T} = \bigvee_{n=1}^{\infty} \mathfrak{T}_{R/I_n}$. By Proposition 3.7, $\rho(M) = \sum_{n=1}^{\infty} \mathfrak{T}_{R/I_n}(M)$ is a torsion preradical which generates $\mathfrak{T}$. Since $\mathfrak{T}_{R/I_n}(R) = 0$ for all n, $\mathfrak{T}(R) = 0$. Since $\bigcap_{n=1}^{\infty} I_n = 0$, $\mathfrak{T}$ is not a TTF class. Assume that $\mathfrak{T}$ has a separating subbasis of one element, J. $\rho^J(R/I_n) = 0$ unless $J \subseteq I_n$, because I_n is a prime ideal. But $\mathcal{T}_{R/J}(R/I_n) = R/I_n$, so that $J \subseteq I_n$ for all n. Thus, $J = 0$, which contradicts the fact that $\mathcal{T}(R) = 0$. Since R_M is a local ring, Lemma 3.5 implies no separating subbasis could have more than one element in it.

Proposition 3.10. If P is a separating subbasis for a torsion class, $\mathcal{T}$, then $\mathcal{T} = \{M \mid$ every non-zero homomorphic image of M has a non-zero I-uniformly negligible module for some $I \in P\}$.

Proof. Let $\mathcal{G} = \{M \in R\text{-mod} \mid M$ is I-uniformly negligible for some $I \in P_1\}$. $\mathcal{G}$ is closed under submodules and homomorphic images. Proposition 3.10 follows from [17, Proposition 2.9]. □

The final results in this chapter show that $\{\mathcal{L}, \vee, \wedge\}$ is distributive and determines how complements in this lattice act.

Proposition 3.11. $\{\mathcal{L}, \vee, \wedge\}$ is distributive.

Proof. It suffices to show that, for any torsion classes $\mathcal{T}$, $\mathcal{U}$, and $\mathcal{V}$, $\mathcal{T} \wedge (\mathcal{U} \vee \mathcal{V}) \subseteq (\mathcal{T} \wedge \mathcal{U}) \vee (\mathcal{T} \wedge \mathcal{V})$, because the reverse inclusion holds in any lattice. Let $\rho(M) = (\pi_M^{-1})(\mathcal{U}[M/\mathcal{V}(M)])$, where π_M is the natural map from M onto $M/\mathcal{V}(M)$. Define ρ_α in the canonical manner used in constructing $\mathcal{U} \vee \mathcal{V}$. Let $M \in \mathcal{T} \wedge (\mathcal{U} \vee \mathcal{V}) = \mathcal{T} \cap (\mathcal{U} \vee \mathcal{V})$. The proof will follow by a transfinite induction on $\lambda(M)$, where $\lambda(M)$ is the smallest ordinal such that $\rho_{\lambda(M)}(M) = \rho_{\lambda(M)+1}(M)$. If $M = \rho_{\lambda(M)}(M)$, then say that M has ρ-length $\lambda(M)$. Assume that $M \in \mathcal{T}$ and that M has ρ-length 1. In this case, $M = (\pi_M^{-1})(\mathcal{U}[M/\mathcal{V}(M)])$. Consider the exact sequence

$$0 \to \mathfrak{V}(M) \to M \to M/\mathfrak{V}(M) \to 0.$$

The left side is in $\mathfrak{T} \wedge \mathfrak{V}$, and the right side is in $\mathfrak{T} \wedge \mathfrak{U}$. Hence M must be in $(\mathfrak{T} \wedge \mathfrak{V}) \vee (\mathfrak{T} \wedge \mathfrak{U})$.

Assume that $M \in (\mathfrak{T} \wedge \mathfrak{U}) \vee (\mathfrak{T} \wedge \mathfrak{V})$ for all $M \in \mathfrak{T} \wedge (\mathfrak{U} \vee \mathfrak{V})$, where M has ρ-length less than α. Suppose that M has ρ-length α and that $\alpha = \beta + 1$. $\rho_\beta(M) \in (\mathfrak{T} \wedge \mathfrak{U}) \vee (\mathfrak{T} \wedge \mathfrak{V})$ by the induction hypothesis, and $M/\rho_\beta(M) \in (\mathfrak{T} \wedge \mathfrak{U}) \vee (\mathfrak{T} \wedge \mathfrak{V})$ by the case $\alpha = 1$. Therefore, $M \in (\mathfrak{T} \wedge \mathfrak{U}) \vee (\mathfrak{T} \wedge \mathfrak{V})$, because M is the extension of one element in $(\mathfrak{T} \wedge \mathfrak{U}) \vee (\mathfrak{T} \wedge \mathfrak{V})$ by another. If α is a limit ordinal and M has ρ-length α, then $Rm \in (\mathfrak{T} \wedge \mathfrak{U}) \vee (\mathfrak{T} \wedge \mathfrak{V})$ for all $m \in M$. This is true because $m \in \rho_\alpha(M)$ implies $m \in \rho_\beta(M)$ for some $\beta < \alpha$. Since m was an arbitrary element of M, the proof of the proposition is complete. □

Since $\{\mathcal{L}, \vee, \wedge\}$ is distributive, a torsion class has at most one lattice complement. Recall that $\mathfrak{T}$ has a lattice complement, $\mathfrak{U}$, if and only if $\mathfrak{T} \wedge \mathfrak{U} = \{0\}$ and $\mathfrak{T} \vee \mathfrak{U} = R\text{-mod}$. When does a torsion class have a complement?

Proposition 3.12. Let $\mathfrak{T}$ and $\mathfrak{T}_1$ be torsion classes on R-mod, and let $\mathfrak{F}$ and $\mathfrak{F}_1$ be the torsion-free classes determined by $\mathfrak{T}$ and $\mathfrak{T}_1$, respectively. Then the following statements are equivalent:

(1) $\mathfrak{T}$ and $\mathfrak{T}_1$ are lattice complements.

(2) $\mathfrak{T} \cap \mathfrak{T}_1 = \{0\}$ and if $\mathfrak{T}(M) + \mathfrak{T}_1(M) = 0$, then $M = 0$.

(3) $\mathfrak{T} \cap \mathfrak{T}_1 = \{0\}$ and $\mathfrak{F} \cap \mathfrak{F}_1 = \{0\}$.

Proof. Since $\mathfrak{T}$ and $\mathfrak{T}_1$ are lattice complements, $\mathfrak{T} \cap \mathfrak{T}_1 = \{0\}$. Define $\rho(M) = (\pi_M^{-1})(\mathfrak{T}[M/\mathfrak{T}_1(M)])$, where π_M is the natural map from M onto $M/\mathfrak{T}_1(M)$. Define ρ_α in the canonical manner used in constructing $\mathfrak{T} \vee \mathfrak{T}_1$. If $\mathfrak{T}(M) + \mathfrak{T}_1(M) = 0$, then $\rho(M) = 0$. Since $\mathfrak{T} \vee \mathfrak{T}_1 = R\text{-mod}$, this implies that $M = 0$.

(2) $\Rightarrow$ (3). $\mathfrak{T} \cap \mathfrak{T}_1 = \{0\}$ by hypothesis. Let $M \in \mathfrak{T} \cap \mathfrak{T}_1$. $\mathfrak{T}(M) + \mathfrak{T}_1(M) = 0$, which implies that $M = 0$.

(3) $\Rightarrow$ (1). $\mathfrak{T} \wedge \mathfrak{T}_1 = \mathfrak{T} \cap \mathfrak{T}_1 = \{0\}$. Let M be any R-module. $M/(\mathfrak{T} \vee \mathfrak{T}_1)(M) \in \mathfrak{T} \cap \mathfrak{T}_1$, which implies that $M = (\mathfrak{T} \vee \mathfrak{T}_1)(M)$. Since M was arbitrary, $\mathfrak{T} \vee \mathfrak{T}_1 = R\text{-mod}$. $\square$

There are rings (such as the integers) for which no non-trivial torsion class has a complement. But what rings have the property that $\{\mathfrak{L}, \vee, \wedge\}$ is complemented?

Proposition 3.13. Let I be a principal lattice ideal (and consequently a lattice in its own right) of $\{\mathfrak{L}, \vee, \wedge\}$ generated by $\mathfrak{T}$. Then I is complemented if and only if $\mathfrak{T}$ is of simple type.

Proof. ($\Rightarrow$). The underlying set for I is the collection of all torsion classes contained in $\mathfrak{T}$. Let $\mathfrak{G}$ be a complete representative set of non-isomorphic simple modules of $\mathfrak{T}$. By hypothesis, $\mathfrak{T}_{\mathfrak{G}}$ has a complement, $\mathfrak{U}$, in $\mathfrak{T}$. But $\mathfrak{U}$ must be zero, because it cannot contain any simple modules. Therefore, $\mathfrak{T}_{\mathfrak{G}} = \mathfrak{T}$, and $\mathfrak{T}$ is of simple type.

($\Leftarrow$). Let $\mathfrak{U} \subset \mathfrak{T}$. $\mathfrak{U}$ is also of simple type. Let $\mathfrak{G}$ be a complete representative set of non-isomorphic simple modules in $\mathfrak{U}$. Define $\mathfrak{V} = \bigvee_{S \in \mathcal{T} \setminus \mathcal{T}_{\mathcal{G}}} \mathcal{T}_S$. Then $\mathfrak{U}$ and $\mathfrak{V}$ are clearly lattice complements in I. □

Definition 3.14. A ring, R, is *semi-artinian* if $\mathcal{S}$ = R-mod.

Corollary 3.15. $\{\mathcal{L},\vee,\wedge\}$ is complemented if and only if R is semi-artinian.

Proof. $\{\mathcal{L},\vee,\wedge\}$ is the principal lattice ideal generated by R-mod. By hypothesis, R-mod is of simple type. By Proposition 3.13, $\{\mathcal{L},\vee,\wedge\}$ is complemented. □

Corollary 3.16. The lattice ideal of $\{\mathcal{L},\vee,\wedge\}$ generated by the simple torsion class is a complemented distributive lattice and hence is a Boolean Algebra.

CHAPTER 4

Decomposition of Torsion Classes

Definition 4.1. Let $\mathfrak{T}$ be a torsion class. $\mathfrak{T}$ admits a *decomposition* if there exist proper subtorsion classes, $\mathfrak{T}_1$ and $\mathfrak{T}_2$, such that $\mathfrak{T}(M) = \mathfrak{T}_1(M) \oplus \mathfrak{T}_2(M)$, for all $M \in$ R-mod.

If a ring satisfies the Primary Decomposition, and if the ring has at least two non-isomorphic simple R-modules, then the simple torsion class admits a decomposition. For the concept of decomposition of torsion classes to be an interesting generalization of the Primary Decomposition, there must be torsion classes which admits decompositions but are not of simple type. Proposition 4.4 leads to just such an example.

Definition 4.2 [3]. A torsion class, $\mathfrak{T}$, is *central splitting* if there is a torsion class, $\mathfrak{U}$, such that $M = \mathfrak{T}(M) \oplus \mathfrak{U}(M)$, for all $M \in$ R-mod.

Proposition 4.3 [13, Theorem 2.4]. A torsion class, $\mathfrak{T}$, is central splitting if and only if there exists a torsion class, $\mathfrak{U}$, such that $R = \mathfrak{T}(R) \oplus \mathfrak{U}(R)$.

Proposition 4.4. Let $\mathfrak{T}$ be any TTF class. Then the following statements are equivalent:

(1) $\mathfrak{T}$ admits a decomposition.

(2) If I is the minimal ideal in $F(\mathfrak{T})$, then R/I is the direct product of two non-zero rings.

(3) If I is the minimal ideal in $F(\mathfrak{T})$, then there is a central splitting torsion class on R/I-mod.

Proof. (1) ⇒ (2). Assume that $\mathfrak{T}(M) = \mathfrak{T}_1(M) \oplus \mathfrak{T}_2(M)$ for all $M \in$ R-mod. Furthermore, assume that neither $\mathfrak{T}_1$ nor $\mathfrak{T}_2$ is zero. That is, assume that the decomposition is non-trivial. Since $R/I \in \mathfrak{T}$, $R/I = \mathfrak{T}_1(R/I) \oplus \mathfrak{T}_2(R/I) = J/I \oplus K/I$, where J and K are left ideals of R. If $x \in J$ and $r \in R$, then $(I:x)xr \subseteq I$, because I is a two-sided ideal of R. Therefore, $(I:x)(xr + I) = 0$, which implies that $xr + I \in \mathfrak{T}_1(R/I) = J/I$. Hence, J is a two-sided ideal of R. Similarly, one obtains that K is a two-sided ideal of R. Thus, J/I and K/I are two-sided ideals of R/I, and their sum is a ring direct sum.

(2) ⇒ (3). Suppose that $R/I = J/I + K/I$. Since J/I and K/I are idempotent two-sided ideals of R/I, the collection of J/I-bounded (K/I-bounded) modules forms a torsion class, $\mathfrak{T}_1(\mathfrak{T}_2)$ [13, Corollary 2.3]. However, $R/I = \mathfrak{T}_1(R/I) \oplus \mathfrak{T}_2(R/I)$, which implies that R/I is central splitting by Proposition 4.3.

(3) ⇒ (1). Assume that $R/I = \mathfrak{T}_1(R/I) \oplus \mathfrak{T}_2(R/I) = J/I \oplus K/I$, where J/I and K/I are idempotent two-sided ideals of R/I. Then J and K are idempotent two-sided ideals of R. Let $\mathfrak{U}_1$ be the class of J-bounded modules, and let $\mathfrak{U}_2$ be the class of K-bounded modules. $\mathfrak{U}_1$ and $\mathfrak{U}_2$ are TTF-classes with minimal filter ideals J and K respectively [13, Corollary 2.3]. Let $M \in \mathfrak{T}$. Then

$$M = RM = (J + K)M \subseteq JM + KM.$$

However, $KJ = JK \subseteq J \cap K = I$. Therefore, $JM \in \mathfrak{U}_2$ and $IM \in \mathfrak{U}_1$. So, for any module, M, $M = \mathfrak{U}_1(M) + \mathfrak{U}_2(M)$. All that remains is to show that the sum is direct. If $M \in \mathfrak{U}_1 \cap \mathfrak{U}_2$, then $JM = 0 = KM$, which implies that $(J + K)M = RM = 0$. □

Example. The following is an example of a torsion class which admits a decomposition but which is not of simple type. Let $R = \prod_{i=1}^{\infty} F_i$, where each F_i is an isomorphic copy of a fixed field, F. Let $I_1 = \{(x_n) \in R \mid x_{2n+1} = 0$ for all positive integers, $n\}$, and let $I_2 = \{(x_n) \in R \mid x_{2n} = 0$ for all positive integers, n, not of the form 2^m for any positive integer, $m\}$. If $I = I_1 \cap I_2$, then I, I_1, and I_2 are idempotent two-sided ideals of R. Consequently, the class of I-bounded modules is a TTF class with I as its minimal filter ideal. $R/I = I_1/I \oplus I_2/I$. By Proposition 4.4, the class of I-bounded modules, which will be denoted by $\mathfrak{J}$, admits a decomposition. All that remains is to show that $\mathfrak{J}$ is not of simple type.

Suppose that $\mathfrak{J}$ is of simple type. Then since R/I is torsion, $R/I + (\bigoplus_{i=1}^{\infty} F_i)$ is also torsion. However, $R/I + \bigoplus_{i=1}^{\infty} F_i$ has zero socle. This implies that $R = I + \bigoplus_{i=1}^{\infty} F_i$, which is not true. Hence, $\mathfrak{J}$ is not of simple type.

The largest part of the remainder of this section is dedicated to proving a theorem which gives necessary and sufficient conditions on a separating subbasis of a torsion class for the torsion class to admit a decomposition. Several lemmas will be needed for the proof of the theorem.

Lemma 4.5. Let $\mathfrak{T}$ and $\mathfrak{T}_1$ be torsion classes such that $\mathfrak{T} \cap \mathfrak{T}_1 = \{0\}$. If $M/N = (A/N) \oplus (B/N) = \mathfrak{T}(M/N) \oplus \mathfrak{T}_1(M/N)$, and if $B = N \oplus N_1 = \mathfrak{T}(B) \oplus \mathfrak{T}_1(B)$, then $M = N_1 \oplus \pi^{-1}(A/N)$, where π is the natural map from M onto M/N.

Proof. $\pi^{-1}(A/N)$ is an extension of N by A/N, both of which are in $\mathfrak{T}$. Thus, $\pi^{-1}(A/N)$ is also in $\mathfrak{T}$. By hypothesis, $N_1 \cap \pi^{-1}(A/N) = 0$. Since $\pi(N_1) = B/N$, $M = N_1 \oplus \pi^{-1}(A/N)$ [5, Lemma 2.5]. □

Definition 4.6. Let M be an R-module. If $\mathcal{C}$ is an ordinal number, and if $\{M_\alpha\}_{\alpha \in \mathcal{C}}$ is a collection of submodules of M, then the collection is called a C-chain for M if the following conditions hold:

(1) $M = \bigcup_{\alpha \in \mathcal{C}} M_\alpha$.

(2) If $\alpha < \beta$, then $M_\alpha \subseteq M_\beta$. Moreover, $M_\alpha = M_\beta$ if and only if $M_\alpha = M$.

(3) $M_{\alpha+1}/M_\alpha$ is a direct sum of cyclic modules for all $\alpha + 1 \in \mathcal{C}$.

Proposition 4.7. Every R-module has at least one C-chain.

Proof. For any non-zero R-module, M, let M_1 be any non-zero cyclic submodule of M. Such a module must exist as long as $M \neq 0$. Assume that M_β is defined for all $\beta < \alpha$, where α is some ordinal number. Whenever M_β is defined, let π_β be the natural map from M onto M_β. If $\alpha = \beta + 1$, then define M_α to be the inverse image under π_β of any non-zero cyclic submodule of M/M_β. If α is a limit ordinal, then define M_α to be $\bigcup_{\beta<\alpha} M_\beta$. There exists an ordinal number α such that $M_\alpha = M$. Let α_o be the smallest such ordinal number. It is clear that $\{M_\beta\}_{\beta \leq \alpha}$ is a C-chain for M. □

The manner of constructing a C-chain is far from unique. However, there is some way of associating modules and their C-chains.

Definition 4.8. For any module, M, let $\alpha(M)$ be the smallest ordinal number such that there exists a collection of submodules of M, $\{M_\beta\}_{\beta\in\alpha(M)}$, which is a C-chain for M. Then M is said to have C-length $\alpha(M)$.

Lemma 4.9. Let $\mathfrak{J}$ and $\mathfrak{J}_1$ be torsion classes on R-mod such that $\mathfrak{J} \cap \mathfrak{J}_1 = \{0\}$. If $0 \to N \to M \to M/N \to 0$ is an exact sequence with $N \in \mathfrak{J}$ and $M/N \in \mathfrak{J}_1$, then $N = \mathfrak{J}(M)$.

Proof. Since $N \in \mathfrak{J}$, $N \subseteq \mathfrak{J}(M)$. For all $m \in \mathfrak{J}(M)$, $(N:m) \in F(\mathfrak{J}_1)$. By Lemma 3.5, $(N:m) = (N:m) + (0:m) = R$. Therefore, $Rm \subseteq N$ and the proof is complete. □

Lemma 4.10. Let G be any set, and let $\{\rho_\alpha\}_{\alpha\in G}$ be a collection of torsion preradicals on R-mod. For all $\alpha \in G$, let $\mathfrak{J}_\alpha$ be the torsion class generated by ρ_α. Suppose that $\sum_{\alpha\in G} \mathfrak{J}_\alpha(M)$ is a direct sum for all $M \in$ R-mod. Then $\rho(M) = \sum_{\alpha\in G} \rho_\alpha(M)$ is a torsion preradical. Moreover, the smallest class containing the class of ρ-modules is $\bigvee_{\alpha\in G} \mathfrak{J}_\alpha$.

Proof. First note that $\sum_{\alpha\in\mathcal{G}} \rho_\alpha(M)$ is, indeed, a direct sum. Let M be an R-module. Then

$$\rho(\rho(M)) = \bigoplus_{\alpha\in\mathcal{G}} \rho_\alpha(\rho(M)) = \bigoplus_{\alpha\in\mathcal{G}} \rho_\alpha(M) = \rho(M).$$

Assume that M is an R-module and that $N \subseteq M$. Then

$$\rho(N) = \bigoplus_{\alpha\in\mathcal{G}} \rho_\alpha(N) = \bigoplus_{\alpha\in\mathcal{G}} (\rho_\alpha(M) \cap N) \subseteq \bigoplus_{\alpha\in\mathcal{G}} \rho_\alpha(M) = \rho(M).$$

By Proposition 2.10, it suffices to show that the class of ρ-modules is closed under submodules and homomorphic images.

Suppose that $M = \rho(M)$, and let $N \subseteq M$. If $m \in N$, then $m = m_1 = m_2 + \ldots + m_k$, where $m_i \in \rho_{\alpha_i}(M)$ and the α_i are distinct. A proof by induction on K will show that $m_1 \in N$. Notice that when this claim is established, the fact that the class of ρ-modules is closed under submodules will also be established. If $K = 1$, then the assertion is trivial.

Assume that whenever $m_1 + m_2 + \ldots + m_k \in N$, where each $m_i \in \rho_{\alpha_i}(M)$, then $m_1 \in N$. Let $m_1 + m_2 + \ldots + m_{k+1} \in N$, where each $m_{\alpha_i} \in \rho_{\alpha_i}(M)$. There exist $I \in F(\mathcal{J}_{\alpha_1})$ and $J \in F(\mathcal{J}_{\alpha_{k+1}})$ such that $Im_1 = 0 = Jm_{k+1}$. Lemma 3.5 yields that $I + J = R$. Let $1 = a_1 + a_{k+1}$, where $a_1 \in I$ and $a_{k+1} \in J$. Then $a_{k+1}(\sum_{i=1}^{k+1} m_i) = \sum_{i=1}^{k} a_{k+1} m_i \in N$. By the induction hypothesis, $a_{k+1} m_1 \in N$. However, $m_1 = (a_1 + a_{k+1})m_1 = a_{k+1} m_1$. Hence, the class of ρ-modules is closed under submodules.

Suppose that $M = \rho(M)$, and let $N \subseteq M$. Since torsion preradicals are closed under homomorphic images, $(\rho_\alpha(M) + N)/N \subseteq \rho_\alpha(M/N)$, for all ordinal numbers α. Since $M = \sum_{\alpha\in\mathcal{G}} \rho_\alpha(M)$, the assertion follows.

Finally, let $\mathfrak{J}$ be the torsion class generated by ρ. Since every ρ-module is contained in $\bigvee_{\alpha\in G} \mathfrak{J}_\alpha$, $\mathfrak{J} \subseteq \bigvee_{\alpha\in G} \mathfrak{J}_\alpha$. Conversely, since $\mathfrak{J}_\alpha \subseteq \mathfrak{J}$ for all $\alpha \in G$, $\bigvee_{\alpha\in G} \mathfrak{J}_\alpha \subseteq \mathfrak{J}$. □

Lemma 4.11. Let $\mathfrak{J}$ be a torsion class with a separating subbasis, P. If $\{P_1,P_2\}$ is a partition of P, and if $\mathfrak{J}_i = \bigvee_{I\in P_i} \mathfrak{J}_{R/I}$ for i = 1,2, then $\mathfrak{J}_1 \cap \mathfrak{J}_2 = \{0\}$.

Proof. Define $\rho(M) = \bigoplus_{I\in P_2} \rho^I(M)$, where ρ^I is the I-uniformly negligible submodule of M. Lemma 3.8 implies that each ρ^I is a torsion preradical. Lemma 4.10 and the fact that P is a separating subbasis imply that ρ is a torsion preradical. $\mathfrak{J}_2$ is the torsion class generated by ρ. For all R-modules, M, and left ideals, I, of P_2, $\rho^I(M) \cap \mathfrak{J}_1(M) = \{0\}$. Therefore, $\rho(M) \cap \mathfrak{J}_1(M) = \{0\}$, which implies that $\mathfrak{J}_1 \cap \mathfrak{J}_2 = \{0\}$.

Lemma 4.12. Let ρ be a torsion preradical, and let $\mathfrak{J}_2$ be the torsion class generated by ρ. Let $\mathfrak{J}_1$ be a torsion class such that $\rho(M) \cap \mathfrak{J}_1(M) = 0$ for all $M \in$ R-mod. If Ext(M,T) = 0 for all $M = \rho(M)$ and $T = \mathfrak{J}_1(T)$, then Ext(M,T) = 0 for all $M = \mathfrak{J}_2(M)$ and $T = \mathfrak{J}_1(T)$.

Proof. The proof will follow by induction on the ρ-length of M. Recall [10] that a $\mathfrak{J}_2$-module has ρ-length α if α is the smallest ordinal number such that $M = \rho_\alpha(M)$. If M has ρ-length 1, then the assertion follows from the hypothesis. Assume that Ext(M,T) = 0 for all $T \in \mathfrak{J}_1$ and $M = \rho_\beta(M)$, where $\beta < \alpha$.

Suppose that $\alpha = \beta + 1$, and $M = \rho_\alpha(M)$. Consider any exact sequence of the form

$$0 \to T \to X \xrightarrow{f} M \to 0,$$

where $T \in \mathfrak{T}_1$. This sequence induces the exact sequence

$$0 \to T \to f^{-1}(\rho_\beta(M)) \xrightarrow{f} \rho_\beta(M) \to 0.$$

The induction hypothesis applies in this case; so the sequence splits. Therefore, $f^{-1}(\rho_\beta(M)) = T \oplus N$, where $N \cong \rho_\beta(M)$. Now consider the exact sequence

$$0 \to T \oplus N/N \to X/N \xrightarrow{f'} M/\rho_\beta(M) \to 0,$$

where $f'(m + N) = f(m) + \rho_\beta(M)$. This sequence satisfies the conditions of the induction when $\alpha = 1$. Thus, the sequence splits and $X/N = (T \oplus N)/N \oplus K/N$, where $K/N \cong M/\rho_\beta(M)$. The hypotheses for Lemma 4.5 are satisfied. If π is the natural map from X onto X/N, then $X = T \oplus \pi^{-1}(K/N)$. Since $M \cong \pi^{-1}(K/N)$, the original sequence splits. Therefore, $\operatorname{Ext}(M,T) = 0$.

Suppose that α is a limit ordinal. Consider an exact sequence of the form

$$0 \to T \to X \xrightarrow{f} M \to 0,$$

where M has ρ-length α and $T \in \mathfrak{T}_1$. For any ordinal number $\beta < \alpha$, there is an induced exact sequence

$$0 \to T \to f^{-1}(\rho_\beta(M)) \to \rho_\beta(M) \to 0.$$

This sequence splits by the induction hypothesis. Therefore, $f^{-1}(\rho_\beta(M)) \subseteq T \oplus \mathfrak{T}_2(X)$. Since $X = \bigcup_{\beta<\alpha} f^{-1}(\rho_\beta(M))$, $X = T \oplus \mathfrak{T}_2(X)$. Thus, the original sequence splits and the lemma is established. □

Lemma 4.13. Let $\mathfrak{T}$ be a torsion class with separating subbasis, P, and let $\{P_1,P_2\}$ be a partition of P. Let $\mathfrak{T}_i = \bigvee_{I \in P_i} \mathfrak{T}_{R/I}$, where $i = 1, 2$, and let $C_i = \{J \mid R/J$ is I-uniformly negligible for some $I \in P_i\}$. Then the following statements are equivalent:

(1) $\mathcal{T}(M) = \mathcal{T}_1(M) \oplus \mathcal{T}_2(M)$ for all $M \in R$-mod.

(2) $\mathrm{Ext}(R/J, M_i) = 0$, whenever $J \in P_j$ and $M_i \in \mathcal{T}_i$, where $i \neq j$.

Proof. (1) $\Rightarrow$ (2). Consider an exact sequence of the form

$$0 \to M_1 \to X \to R/J \to 0,$$

where $M_1 \in \mathcal{T}_1$ and $J \in P_2$. X is the extension on one member of $\mathcal{T}$ by another. Therefore, X is in $\mathcal{T}$, and the hypothesis implies that $X = \mathcal{T}_1(X) \oplus \mathcal{T}_2(X)$. Lemma 4.9 implies that $M_1 = \mathcal{T}_1(X)$. Hence, the sequence splits and $\mathrm{Ext}(R/J, M_1) = 0$. A similar argument yields the second part of condition (2).

(2) $\Rightarrow$ (1). Let $\rho(M) = \bigoplus_{I \in P_2} \rho^I(M)$, where $\rho^I(M)$ is the I-uniformly negligible submodule of M. The sum is direct because P is a separating subbasis of $\mathcal{T}$. Lemma 4.10 implies that ρ is a torsion preradical which generates $\mathcal{T}_2$.

Assume that $T = \rho^I(T)$ for some $I \in P_2$, and that $M \in \mathcal{T}_1$. Consider any exact sequence of the form

$$0 \to M \to X \xrightarrow{f} T \to 0.$$

For all $x \in T$, there is an induced exact sequence

$$0 \to M \to f^{-1}(Rx) \xrightarrow{f} Rx \to 0.$$

Since $Rx \cong R/J$ for some $J \in C_2$, condition (2) implies that the sequence splits. Thus, $f^{-1}(Rx) \subseteq \mathcal{T}_1(X) \oplus \mathcal{T}_2(X)$, because Lemma 4.9 implies that $M = \mathcal{T}_1(X)$. Since $X = \bigcup_{x \in T} f^{-1}(Rx)$, $X = \mathcal{T}_1(X) \oplus \mathcal{T}_2(X) = M \oplus \mathcal{T}_2(X)$. Hence $\mathrm{Ext}(T,M) = 0$.

Now let $T = \rho(T)$ and let $M \in \mathfrak{T}_1$. $\operatorname{Ext}(T,M) = $ $= \operatorname{Ext}(\bigoplus_{I \in P_2} \rho^I(T),M) \cong \prod_{I \in P_2} \operatorname{Ext}(\rho^I(T),M) = 0$. Hence, by Lemma 4.12, $\operatorname{Ext}(T,M) = 0$ for all $T \in \mathfrak{T}_2$ and $M \in \mathfrak{T}_1$.

To complete the proof, let $m \in \pi_X^{-1}(\mathfrak{T}_2[X/\mathfrak{T}_1(X)])$ where π_X is the natural map from X onto $X/\mathfrak{T}_1(X)$, and X is any module. There exists an exact sequence

$$0 \to \mathfrak{T}_1(X) \to Rm + \mathfrak{T}_1(X) \to \frac{Rm + \mathfrak{T}_1(X)}{\mathfrak{T}_1(X)} \to 0.$$

This sequence splits, so that $\pi_X^{-1}(\mathfrak{T}_2[X/\mathfrak{T}_1(X)]) = \mathfrak{T}_1(X) \oplus \mathfrak{T}_2(X)$. A similar argument yields the analagous equality with the roles of $\mathfrak{T}_1$ and $\mathfrak{T}_2$ inter-changed. Proposition 3.6 implies that $(\mathfrak{T}_1 \vee \mathfrak{T}_2)(M) = \mathfrak{T}_1(M) \oplus \mathfrak{T}_2(M)$ for all $M \in$ R-mod. Since $\mathfrak{T} = \mathfrak{T}_1 \vee \mathfrak{T}_2$, the lemma is established. □

Lemma 4.14. Let $\mathfrak{T}$ be a torsion class with a separating subbasis, P. Let $\{P_1,P_2\}$ be a partition of P with $\mathfrak{T}_i$ and C_i defined as in Lemma 4.13. If $\mathfrak{T}_2([\prod R/J]/\oplus R/J)_{J \in C_1} = 0$, then

$\mathfrak{T}_2([\prod \mathfrak{T}_{R/I}(M)]/\oplus \mathfrak{T}_{R/I}(M))_{I \in P_1} = 0$, for all $M \in$ R-mod.

Proof. If M is an R-module, then let $M^* = ([\prod_{I \in P_1} \mathfrak{T}_{R/I}(M)]/ \bigoplus_{I \in P_1} \mathfrak{T}_{R/I}(M))$. Define $\rho(M) = \bigoplus_{I \in P_2} \rho^I(M)$. Then Lemma 4.10 implies that ρ is a torsion preradical which generates $\mathfrak{T}_2$.

Let M be an R-module, and let $x = (x_I) + \bigoplus_{I \in P_1} \mathfrak{T}_{R/I}(M) \in \mathfrak{T}_2(M^*)$. Without loss of generality, assume that $x \in \rho^I(M^*)$

for some $I \in P_2$. $Rx \cong R/J$, where $J \in \mathcal{C}_2$. For all $K \in P_1$, define $\beta(K)$ to be the smallest ordinal number such that $x_K \in (\rho^K)_{\beta(K)}(M)$. For all K, $\beta(K) = \alpha(K) + 1$ is not a limit ordinal.

For each $K \in P_1$, define the submodule of M, $B_K = (\rho^K)_{\alpha(K)}(M)$. For all $K \in P_1$, $x_K + B_K \in \rho^K(\mathcal{T}_{R/K}(M)/B_K)$. Whenever $x_K \neq 0$, there exists a homomorphism $f_K \in \mathrm{Hom}([Rx + B_K]/B_K, R/J_K)$, where R/J_K is K-uniformly negligible, and $f_K(x_K + B_K) \neq 0$. Let $y = (f_K(x_K + B_K)) + \bigoplus\limits_{J \in \mathcal{C}_1} R/J \in (\pi\ R/J/\oplus\ R/J)_{J \in \mathcal{C}_1}$. Let $r \in J$. Since $rx = 0$, r annihilates all but a finite number of x_K. Consequently, $Jy = 0$. This implies that $y = 0$, because $J \in F(\mathcal{T}_2)$. Consequently, $x = 0$ and the lemma is established. □

Theorem 4.15. Let $\mathcal{T}$ be a torsion class with a separating subbasis, P. Let $\{P_1, P_2\}$ be a partition of P, and define $\mathcal{T}_i$ and $\mathcal{C}_i$ as in Lemma 4.13 for $i = 1, 2$. Then the following conditions are equivalent:

(1) $\mathcal{T}(M) = \mathcal{T}_1(M) \oplus \mathcal{T}_2(M) = 0$, for all $M \in R\text{-mod}$.

(2) (a) $\mathcal{T}_1(\pi\ R/J/\oplus\ R/J) = 0 = \mathcal{T}_2(\pi\ R/J/\oplus\ R/J)$, where the product and sum run over $\mathcal{C}_2$ and $\mathcal{C}_1$ respectively.

(b) $\mathrm{Ext}(R/I, R/J) = 0 = \mathrm{Ext}(R/J, R/I)$ for all $I \in \mathcal{C}_1$ and $J \in \mathcal{C}_2$.

(c) If $\mathcal{Q}$ is any set, then $\mathcal{T}_i(\pi\ R/J_\alpha/\oplus\ R/J_\alpha) = 0$, where $\alpha \in \mathcal{Q}$ and each R/J_α is I-uniformly negligible for a fixed $I \in P_j$, $i \neq j$.

(d) Whenever $0 \to \mathfrak{T}_i(Rx) \to Rx \to R/J \to 0$ is an exact sequence with $J \in C_j$, $i \neq j$, then $\mathfrak{T}_i(Rx)$ has non-limit ordinal ρ^i-length, where $\rho^i(M) = \bigoplus_{I \in P_i} \rho^I(M)$.

(e) Let $0 \to B \to Rx \to R/J \to 0$ be an exact sequence with B I-uniformly negligible for some $I \in P_i$, and $J \in C_j$, where $i \neq j$. Then B has a C-chain of finite length.

Proof. Define ρ^1 and ρ^2 as in condition (d) of the hypothesis. Lemma 4.10 and the fact that P is a separating subbasis for $\mathfrak{T}$ imply that ρ^1 and ρ^2 are torsion preradicals which generate $\mathfrak{T}_1$ and $\mathfrak{T}_2$, respectively. These definition will hold throughout the remainder of this proof.

(1) ⇒ (2). (a) Assume that there exists a submodule, M, of $\prod_{J \in C_2} R/J$ such that $(M + \bigoplus_{J \in C_2} R/J)/\bigoplus_{J \in C_2} R/J \in \mathfrak{T}_1$. Without loss of generality, one may obtain a submodule, $N/\bigoplus_{J \in C_2} R/J$, of $(M + \bigoplus_{J \in C_2} R/J)/\bigoplus_{J \in C_2} R/J$ which is isomorphic to R/K, for some $K \in C_1$. Consider the exact sequence

$$0 \to \bigoplus_{J \in C_2} R/J \to N \to R/K \to 0.$$

Since the left end of the sequence is in $\mathfrak{T}_2$, Lemma 4.13 implies that the sequence splits, and that R/K is isomorphic to a submodule of $\prod_{J \in C_2} R/J$. However, this is a contradiction because $\mathrm{Hom}(R/K, \prod_{J \in C_2} R/J) \cong \prod_{J \in C_2} \mathrm{Hom}(R/K, R/J) = 0$. The argument for the other equality is entirely analagous.

(b) This condition follows immediately from Lemma 4.13.

(c) Let $\mathfrak{a}$ be any set, and let $I \in P_2$. Let $M_o = \prod_{\alpha \in \mathfrak{a}} R/J_\alpha$, and let $N_o = \bigoplus_{\alpha \in \mathfrak{a}} R/J_\alpha$, where R/J_α is I-uniformly negligible, for all $\alpha \in \mathfrak{a}$. Let $L \in \mathcal{C}_1$. Then the exact sequence

$$0 \to N_o \to M_o \to M_o/N_o \to 0$$

yields the exact sequence

$$\mathrm{Hom}(R/L, M_o) \to \mathrm{Hom}(R/L, M_o/N_o) \to \mathrm{Ext}(R/L, N_o).$$

However, $\mathrm{Hom}(R/L, M_o) \cong \prod_{\alpha \in \mathfrak{a}} \mathrm{Hom}(R/L, R/J_\alpha) = 0$. By Lemma 4.13, $\mathrm{Ext}(R/L, N_o) = 0$. Therefore, $\rho^1(M_o/N_o) = 0$, which implies that $\mathfrak{J}_1(M_o/N_o) = 0$. The argument for the other equality is analagous.

(d) Let $0 \to \mathfrak{J}_1(Rx) \to Rx \to R/J \to 0$, where $J \in \mathcal{C}_2$. By Lemma 4.13, the sequence splits. Therefore, $\mathfrak{J}_1(Rx)$ is cyclic and cannot have limit ordinal ρ^1-length.

(e) Let $0 \to B \to Rx \to R/J \to 0$ be an exact sequence, where B is I-uniformly negligible for some $I \in P_1$, and where $J \in \mathcal{C}_2$. By Lemma 4.13, this sequence splits and B is cyclic. Therefore, B has C-length 1.

(2) ⇒ (1). Let $M \in \mathfrak{J}$. By Lemmas 4.9 and 4.13 it is sufficient to show that $\mathrm{Ext}(R/J, \mathfrak{J}_1(M)) = 0$, for all $J \in \mathcal{C}_2$. First, assume that $\mathfrak{J}_1(M)$ has ρ^1-length 1. Then $\mathfrak{J}_1(M) = \bigoplus_{I \in P_1} \rho^I(M)$. Let $N = \prod_{I \in P_1} \rho^I(M)$. For all $J \in \mathcal{C}_2$, the exact sequence

$$0 \to \mathfrak{J}_1(M) \to N \to N/\mathfrak{J}_1(M) \to 0$$

induces an exact sequence

$$\mathrm{Hom}(R/J, N/\mathfrak{J}_1(M)) \to \mathrm{Ext}(R/J, \mathfrak{J}_1(M)) \to \mathrm{Ext}(R/J, N).$$

Condition (a) and Lemma 4.14 imply that $\mathrm{Hom}(R/J, N/\mathfrak{J}_1(M)) = 0$. On the other hand, $\mathrm{Ext}(R/J, N) \cong \prod_{I \in P_1} \mathrm{Ext}(R/J, \wp^I(M))$. The case when $\mathfrak{J}_1(M)$ has $\wp^1$-length 1 will will be complete if $\mathrm{Ext}(R/J, B) = 0$, where $J \in \mathcal{C}_2$, and $B = \wp^I(B)$ for some $I \in P_1$. The proof will follow by induction on the $\mathcal{C}$-length of B.

Suppose that B has $\mathcal{C}$-length 1; i.e. B is a direct sum of cyclic modules. Then $B \cong \bigoplus_{\delta \in \mathcal{D}} R/J_\delta$, where each R/J_δ is I-uniformly negligible for some fixed $I \in P_1$. Let $J \in \mathcal{C}_2$, and let $M = \prod_{\delta \in \mathcal{D}} R/J_\delta$. The exact sequence

$$0 \to B \to M \to M/B \to 0$$

induces an exact sequence

$$\mathrm{Hom}(R/J, M/B) \to \mathrm{Ext}(R/J, B) \to \mathrm{Ext}(R/J, M).$$

Condition (c) implies that $\mathrm{Hom}(R/J, M/B) = 0$. $\mathrm{Ext}(R/J, M) \cong \prod_{\delta \in \mathcal{D}} (R/J, R/J_\delta) = 0$, by condition (b). Thus, $\mathrm{Ext}(R/J, B) = 0$, and the first step in the induction is complete. Assume that whenever B has $\mathcal{C}$-length less than α, $\mathrm{Ext}(R/J, B) = 0$, for all $J \in \mathcal{C}_2$ and B I-uniformly negligible for some $I \in P_1$. Assume that B has $\mathcal{C}$-length α.

If $\alpha = \beta + 1$, then let $\{B_\delta\}_{\delta \leq \alpha}$ be a $\mathcal{C}$-chain for B. For any exact sequence

$$0 \to B \to X \xrightarrow{f} R/J \to 0,$$

where $J \in \mathcal{C}_2$, induces an exact sequence

$$0 \to B/B_\beta \to X/B_\beta \xrightarrow{f'} R/J \to 0,$$

where $f'(x + B_\beta) = f(x)$. Since B/B_β has $\mathcal{C}$-length 1, the induction hypothesis implies that this sequence splits. Therefore,

$X/B_\beta = B/B_\beta \oplus H/B_\beta$, where $H/B_\beta \cong R/J$. Since B_β has C-length β, the induction hypothesis implies that the exact sequence

$$0 \to B_\beta \to H \to R/J \to 0$$

splits. $H = B_\beta \oplus H_1$, where $H_1 \cong R/J$. The hypotheses of Lemma 4.5 are satisfied. Hence, $X = B \oplus H_1$, and the original sequence splits.

Now assume that α is a limit ordinal, and that B has C-length α. Assume that

$$0 \to B \to X \to R/J \to 0$$

is an exact sequence with B having C-length α. For $x \in X \backslash B$, there is an induced exact sequence

$$0 \to B \cap Rx \to Rx \to R/(B:x) \to 0.$$

Condition (e) implies that $B \cap Rx$ has finite C-length. By the induction hypothesis, this sequence splits. Therefore, $Rx \subseteq \mathcal{T}_1(X) \oplus \mathcal{T}_2(X)$ for all $x \in X\backslash B$. However, $B = \mathcal{T}_1(X)$ by Lemma 4.9. Hence, $X = \mathcal{T}_1(X) \oplus \mathcal{T}_2(X) = B \oplus \mathcal{T}_2(X)$, and the original sequence splits. This completes the induction hypothesis, and consequently, the case for $\mathcal{T}_1(N)$ having ρ^1-length 1 is finished. Assume that whenever $\mathcal{T}_1(M)$ has ρ^1-length less than α, then $\text{Ext}(R/J, \mathcal{T}_1(M)) = 0$ for all $J \in C_2$. Suppose that $\mathcal{T}_1(M)$ has ρ^1-length α.

If $\alpha = \beta + 1$, then the induction hypothesis yields that $\text{Ext}(R/J, (\rho^1)_\beta(M)) = 0 = \text{Ext}(R/J, \mathcal{T}_1(M)/(\rho^1)_\beta(M))$. Consequently, $\text{Ext}(R/J, M) = 0$.

Finally, assume that α is a limit ordinal, and that $\mathcal{T}_1(M)$ has ρ^1-length α. Suppose that there is an exact sequence

$$0 \to M \to X \xrightarrow{f} R/J \to 0.$$

Let $x \in X$ such that $f(x) = r + J \neq 0$. There is an exact sequence

$$0 \to M \cap Rx \to Rx \to R/(J:r) \to 0.$$

Lemma 4.9 implies that $M \cap Rx = \mathcal{T}_1(Rx)$. Condition (d) implies that $M \cap Rx$ does not have limit ordinal ρ^1-length. Since $(\rho^1)_\alpha$ is a torsion preradical, the ρ^1-length of $M \cap Rx$ is strictly less than α. The induction hypothesis applies and $Rx \subseteq \mathcal{T}_1(X) \oplus \mathcal{T}_2(X)$. If $f(x) = 0$, then $x \in M \subseteq \mathcal{T}_1(X)$. Consequently, $X = \mathcal{T}_1(X) \oplus \mathcal{T}_2(X)$. Lemma 4.9 implies that $M = \mathcal{T}_1(X)$. The original sequence splits and $\operatorname{Ext}(R/J, \mathcal{T}_1(M)) = 0$ for all $J \in \mathcal{C}_2$ and $M \in R\text{-mod}$. Lemma 4.13 implies that the theorem is established. □

The next two propositions reduce the problem of when $\mathcal{T}(M) = \bigoplus_{\alpha\in\mathcal{Q}} \mathcal{T}_\alpha(M)$, where $\{\mathcal{T}_\alpha\}_{\alpha\in\mathcal{Q}}$ is a collection of subtorsion classes of $\mathcal{T}$, to the problem handled in Theorem 4.15. The case where $\mathcal{Q}$ is a finite set leads to a particularly nice result and will be done first.

Proposition 4.16. Let $\mathcal{T}$ be a torsion class with a separating subbasis, P. Let $\mathcal{T}_K = \bigvee_{I\in P_k} \mathcal{T}_{R/I}$, where $\{P_1,\ldots,P_n\}$ is a partition of P. Then the following statements are equivalent:

(1) $\mathcal{T}(M) = \bigoplus_{i=1}^{n} \mathcal{T}_i(M)$ for all $M \in R\text{-mod}$.

(2) If $i \neq j$, then $(\mathcal{T}_i \vee \mathcal{T}_j)(M) = \mathcal{T}_i(M) \oplus \mathcal{T}_j(M)$ for all $M \in R\text{-mod}$.

Proof. (1) ⇒ (2). Let $M \in \mathcal{T}_i \vee \mathcal{T}_j$, where $i \neq j$. Condition (1) implies that $M = \bigoplus_{i=1}^{n} \mathcal{T}_i(M)$. If $k \neq i,j$, then $\mathcal{T}_k \cap (\mathcal{T}_i \vee \mathcal{T}_j) = 0$

because P is a separating subbasis of $\mathcal{T}$. Hence, condition (2) is established.

(2) ⇒ (1). Define $\rho(M) = \bigoplus_{i=1}^{n} \mathcal{T}_i(M)$ for all $M \in$ R-mod. Lemma 4.10 implies that ρ is a torsion preradical which generates $\bigvee_{i=1}^{n} \mathcal{T}_i = \mathcal{T}$. Let $M = \rho_2(M)$, and let $m \in \rho_2(M)$. If $m \notin \rho_1(M)$, then one may assume, without loss of generality, that $m + \rho_1(M) \in \mathcal{T}_i(M/\rho(M))$ for some $i = 1,\ldots,n$. Consider the exact sequence

$$(*) \quad 0 \to \bigoplus_{j\neq i} \mathcal{T}_j(M) \to Rm + \rho(M) \to [Rm + \rho(M)]/\bigoplus_{j\neq i} \mathcal{T}_j(M) \to 0.$$

Consider the exact sequence

$$0 \to \rho(M)/\bigoplus_{j\neq i} \mathcal{T}_j(M) \to Rm + \rho(M)/\bigoplus_{j\neq i} \mathcal{T}_j(M) \to Rm + \rho(M)/\rho(M) \to 0.$$

Both ends of this sequence are in $\mathcal{T}_i$. Hence, $Rm + \rho(M)/\bigoplus_{j\neq i} \mathcal{T}_j(M) \in \mathcal{T}_i$. Notice that $\mathrm{Ext}(Rm + \rho(M)/\bigoplus_{j\neq i} \mathcal{T}_j(M), \bigoplus_{j\neq i} \mathcal{T}_j(M)) \cong \bigoplus_{j\neq i} \mathrm{Ext}(Rm + \rho(M)/\bigoplus_{j\neq i} \mathcal{T}_k(M), \mathcal{T}_j(M))$, because the direct sum is finite. However, condition (2) implies that $\mathrm{Ext}(Rm + \rho(M)/\bigoplus_{j\neq i} \mathcal{T}_j(M), \bigoplus_{j\neq i} \mathcal{T}_j(M)) = 0$, and that (*) splits. Therefore, $Rm + \rho(M) \subseteq \bigoplus_{i=1}^{n} \mathcal{T}_i(M) = \rho(M)$. Thus, $\rho_2(M) = \rho(M) = \mathcal{T}(M)$ for all $M \in$ R-mod. □

Proposition 4.17. Let $\mathcal{T}$ be a torsion class with a separating subbasis, P. Let $\{P_\alpha\}_{\alpha\in\mathcal{G}}$ be a partition of P. Let $\mathcal{T}_\alpha = \bigvee_{I\in P_\alpha} \mathcal{T}_{R/I}$. Then the following statements are equivalent:

(1) $\mathcal{T}(M) = \bigoplus_{\alpha\in\mathcal{G}} \mathcal{T}_\alpha(M)$, for all $M \in$ R-mod.

(2) If $\alpha_o \in \mathcal{G}$, then $\mathcal{T}(M) = \mathcal{T}_{\alpha_o}(M) \oplus (\bigvee_{\alpha\neq\alpha_o} \mathcal{T}_\alpha)(M)$, for all $M \in$ R-mod.

Proof. (1) ⇒ (2). This part is immediate, because condition (1) implies that $\mathcal{T}(M) \subseteq \mathcal{T}_{\alpha_0}(M) \oplus (\bigvee_{\alpha\neq\alpha_0} \mathcal{T}_{\alpha})(M)$, for all $M \in R\text{-mod}$.

(2) ⇒ (1). Let $\rho(M) = \bigoplus_{\alpha\in\mathcal{G}} \mathcal{T}_{\alpha}(M)$. Lemma 4.10 implies that ρ is a torsion preradical which generates $\mathcal{T}$. Let $M \in \mathcal{T}$, and let $m \in \pi^{-1}\rho([M/\rho(M)])$, where π is the natural map from M onto $M/\rho(M)$. If $m \notin \rho(M)$, then one may assume that there is an $\alpha_0 \in \mathcal{G}$ such that $m \in \pi^{-1}(\mathcal{T}_{\alpha_0}[M/\rho(M)])$. There exists $J \in F(\mathcal{T}_{\alpha_0})$ such that $Jm \subseteq \rho(M)$. By hypothesis, $m = m_1 + m_2$, where $m_1 \in \mathcal{T}_{\alpha_0}$ and $m_2 \in \bigvee_{\alpha\neq\alpha_0} \mathcal{T}_{\alpha}(M)$. Without loss of generality, assume that $J \subseteq (0:m_1)$. Therefore, $Jm_2 \subseteq \rho(M)$. Lemma 3.5 implies that $J + (0:m_2) = R$. Consequently, $m_2 \in \rho(M)$. Since $m_1 \in \rho(M)$, $m \in \rho(M)$. Therefore, $\mathcal{T}(M) = \rho_2(M) = \rho(M)$, for all $M \in R\text{-mod}$. □

We are now prepared to obtain some corollaries of these results where $\mathcal{T}$ is the simple torsion class.

Corollary 4.18 [2, Theorem 3.1]. Let $\mathcal{S}$ be the simple torsion class. Then the following statements are equivalent:

(1) R satisfies the Primary Decomposition.

(2) (a) $\operatorname{Soc}(\prod_{\alpha\in\mathcal{G}} S_{\alpha} / \bigoplus_{\alpha\in\mathcal{G}} S_{\alpha}) = 0$, where $\{S_{\alpha}\}_{\alpha\in\mathcal{G}}$ is a complete representative set of non-isomorphic simple R-modules.

(b) $\operatorname{Ext}(U,V) = 0$ whenever U and V are non-isomorphic simple R-modules.

(c) If $\mathcal{G}$ is any set, then $\mathcal{T}_U(\prod_{\alpha\in\mathcal{G}} V_\alpha / \bigoplus_{\alpha\in\mathcal{G}} V_\alpha) = 0$ whenever U and V are non-isomorphic simple R-modules and $V_\alpha \cong V$ for all $\alpha \in \mathcal{G}$.

(d) If $L \subsetneq M$ are left ideals of R such that $M/L \in \mathcal{T}_V$ and $R/M \cong U$, then the ρ-length of M/L is not a limit ordinal where $\rho(M) = \mathrm{Soc}(M)$.

Proof. Let $P = \{M_\alpha\}_{\alpha\in\mathcal{G}}$, where $\{R/M_\alpha\}_{\alpha\in\mathcal{G}}$ is a complete representative set of non-isomorphic simple R-module. Then P is a separating subbasis for $\mathcal{S}$.

(1) ⇒ (2). Proposition 4.17 implies that $\mathcal{S}(M) = \mathcal{T}_{R/M_\alpha}(M) \oplus \bigvee_{\beta\neq\alpha} \mathcal{T}_{R/M_\beta}(M)$ for all $M \in$ R-mod and $\alpha \in \mathcal{G}$. Since each S_α is M_α-uniformly negligible, condition (a) follows from condition (a) of Theorem 4.15. Lemma 2.8 implies that every M_α-uniformly negligible module is a direct sum of simples each of which is isomorphic to R/M_α. Hence conditions (b), (c), and (d) follow immediately from conditions (b), (c), and (d) of Theorem 4.15.

(2) ⇒ (1). Let $P_1 = \{M_{\alpha_1}\}$, and let $P_2 = \{M_\alpha\}_{\alpha\in\mathcal{G}\setminus\{\alpha_1\}}$. By Proposition 4.17, it suffices to show that $\mathcal{S}(M) = \mathcal{T}_{R/M_{\alpha_1}}(M) \oplus \bigvee_{\alpha\neq\alpha_1} \mathcal{T}_{R/M_\alpha}(M)$, for all $M \in$ R-mod. Lemma 2.8 implies that every M_α-uniformly negligible module is a direct sum of cyclic R-modules, and hence condition (e) of Theorem 4.15 is satisfied. By Lemma 2.8 and conditions (a) - (d) of this corollary, conditions (a) - (d) of Theorem 4.15 are satisfied. Therefore, the corollary is established. □

Corollary 4.19. For a left Noetherian ring, R, let $\mathfrak{J}$ be a torsion class with separating subbasis, P. Let $\{P_1, P_2\}$ be a partition of P, and let $\mathfrak{J}_i = \bigvee_{i \in P_i} \mathfrak{J}_{R/I}$ for i = 1,2. Then the following statements are equivalent:

(1) $\mathfrak{J}(M) = \mathfrak{J}_1(M) \oplus \mathfrak{J}_2(M)$, for all $M \in$ R-mod.

(2) $\mathrm{Ext}(R/I, R/J) = 0 = \mathrm{Ext}(R/J, R/I)$, for all $J \in \mathcal{C}_1$ and $I \in \mathcal{C}_2$ where $\mathcal{C}_1$ and $\mathcal{C}_2$ are defined as in Lemma 4.13.

Proof. (1) $\Rightarrow$ (2). This follows immediately from Theorem 4.15.

(2) $\Rightarrow$ (1). It is sufficient to show that conditions (a) - (e) of Theorem 4.15 are satisfied. Conditions (d) and (e) follow directly from the fact that R is Noetherian. Assume $\mathfrak{J}_1(M/N) \neq 0$, where $M = \prod_{J \in \mathcal{C}_2} R/J$ and $N = \bigoplus_{J \in \mathcal{C}_2} R/J$. There exist $I \in F(\mathfrak{J}_1)$ and $(x_J) + N \in M/N$ such that $I((x_J) + N) = 0$, or $I(x_J) \subseteq N$. Since R is Noetherian, I is finitely generated. Each generator annihilates all but a finite number of x_J. Therefore, I annihilates all but a finite number of x_J. However, since $Rx_J \in \mathfrak{J}_2$ and $I \in F(\mathfrak{J}_1)$, the only x_J that I could annihilate were zero already. Hence condition (a) is established.

Let $\mathcal{A}$ be any set. Consider a collection, $\{R/J_\alpha\}_{\alpha \in \mathcal{A}}$ of cyclic R-modules which are all I-uniformly negligible for some fixed $I \in \mathcal{P}_2$. Assume that $\mathfrak{J}_1(M/N) \neq 0$, where $M = \prod_{\alpha \in \mathcal{A}} R/J_\alpha$ and $N = \bigoplus_{\alpha \in \mathcal{A}} R/J_\alpha$. There exist $I \in F(\mathfrak{J}_1)$ and $(x_J) + N \in M/N$ such that $I((x_J) + N) = 0$, or $Ix_J \subseteq N$. Since R is Noetherian, I is finitely generated. Each generator annihilates all but finite number of x_J. However, since $Rx_J \in \mathcal{T}_2$, the only x_J that I could annihilate were zero already. Hence, condition (c) is established. Condition (b) is condition (2) of the corollary, and the corollary is established by Theorem 4.15. □

Corollary 4.20 [6, Theorem 1.1]. For a left Noetherian ring, R, the following statements are equivalent:

(1) R satisfies the Primary Decomposition.

(2) Ext(U,V) = 0 whenever U and V are non-isomorphic simple R-modules.

Proof. Let $P = \{M_\alpha\}_{\alpha\in\mathcal{A}}$ be a collection of left ideals of R such that $\{R/M_\alpha\}_{\alpha\in\mathcal{A}}$ is a complete representative set of non-isomorphic simple R-modules. Then P is a separating subbasis for $\mathcal{S}$. Let $M_o \in P$. Consider the partition of P, $\{\{M_o\}P\setminus\{M_o\}\}$.

(1) ⇒ (2). This follows immediately from Corollary 4.18.

(2) ⇒ (1). Corollary 4.19 and the discussion above imply that $\mathcal{S}(M) = \mathcal{T}_{R/M_o}(M) \oplus \bigvee_{M_\alpha \in P\setminus\{M_o\}} \mathcal{T}_{R/M_\alpha}(M)$ for all $M_o \in P$. Proposition 4.17 implies that R satisfies the Primary Decomposition. □

Let $\mathcal{T}$ be a torsion class of simple type. If $\mathcal{T}$ admits a decomposition, then does it necessarily follow that $\mathcal{T}$ has the Primary Decomposition? The following example answers this question in the negative.

Example. Before the example can be stated several lemmas are required.

Lemma 4.21. If R is a Noetherian ring, then the upper triangular matrix ring over R is also Noetherian.

Proof. If R is a Noetherian ring, then the ring $S = \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}$ is also a Noetherian ring. Since the upper triangular matrix ring is a finitely generated module over S, it must be a Noetherian module, and hence a Noetherian ring. □

Lemma 4.22 [2, Corollary 4.3]. Let $U \cong R/M$ and $V \cong R/N$ be non-isomorphic simple R-modules. If M and N are two-sided ideals such that $NM = N \cap M$, then $\text{Ext}(U,V) = 0$.

Now the example can be demonstrated. Consider the Abelian group $G = Z/(2) \oplus Z/(3)$. Make this group into a zero ring, i.e. define $xy = 0$, for all $x,y \in G$. Following McCoy [16, page 8], imbed G into a ring, R, with identity. In this construction, $R = \{(a,b) | a \in G \text{ and } b \in Z\}$. Addition is defined co-ordinate wise, and $(a,b)(x,y) = (bx + ya, by)$. Let $S_1 = \{(a,0) | a \in Z/(2) \oplus 0\}$ and let $S_2 = \{(a,0) | a \in 0 \oplus Z/(3)\}$. Then S_1 and S_2 are non-isomorphic simple submodules of R such that $\text{Ext}(S_1,S_2) = 0 = \text{Ext}(S_2,S_1)$. If M_i is the annihilator of S_i, then $M_1M_2 = M_2M_1 = M_1 \cap M_2$. Hence, R is a commutative Noetherian ring with two non-isomorphic simple R-modules in Soc(R). Furthermore, the product of the annihilators of these simple R-modules equals their intersection.

Let R be as above, and let T be the upper triangular matrix ring over R. Let $A = \begin{pmatrix} M_1 & R \\ 0 & R \end{pmatrix}$, $B = \begin{pmatrix} M_2 & R \\ 0 & R \end{pmatrix}$, $C = \begin{pmatrix} R & R \\ 0 & M_2 \end{pmatrix}$, and $D = \begin{pmatrix} R & R \\ 0 & M_2 \end{pmatrix}$ be maximal left ideals of T which are also two-sided ideals of T. Let $P_1 = \{A,C\}$, and let $P_2 = \{B,D\}$. For all $I \in P_1$ and $J \in P_2$, $IJ = JI = I \cap J$. Lemma 4.22 implies that $\mathrm{Ext}(R/I, R/J) = \mathrm{Ext}(R/J, R/I) = 0$. However, $\mathrm{Ext}(R/A, R/C) \neq 0$. To see this, consider the exact sequence

$$0 \to \begin{pmatrix} 0 & S_1 \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & S_1 \\ 0 & S_1 \end{pmatrix} \to \frac{\begin{pmatrix} 0 & S_1 \\ 0 & S_1 \end{pmatrix}}{\begin{pmatrix} 0 & S_1 \\ 0 & 0 \end{pmatrix}} \to 0.$$

The left side is isomorphic to R/A, and the right side is isomorphic to R/C. If this sequence split, then the middle term of the sequence would be annihilated by $A \cap C$, which it is not. Therefore, $\mathrm{Ext}(R/A, R/C) \neq 0$.

Let $\mathfrak{T}$ be the torsion class of simple type with separating subbasis $P = \{A,B,C,D\}$. By Corollary 4.20, $\mathfrak{T}$ does not satisfy the Primary Decomposition. Let $P_1 = \{A,C\}$ and $P_2 = \{B,D\}$. Corollary 4.19 implies that $\mathfrak{T}$ admits a decomposition.

The results developed in this paper are now used to generalize a theorem due to Albu [1, Theorem 5].

Definition 4.23. Let $\mathfrak{T}$ be a torsion class, and let $\mathfrak{D} = \{\mathfrak{T}_\alpha\}_{\alpha \in \mathfrak{G}}$ be a collection of torsion subclasses of $\mathfrak{T}$ such that $\mathfrak{T} = \bigvee_{\alpha \in \mathfrak{G}} \mathfrak{T}_\alpha$. For $M \in \mathfrak{T}$, define the $\mathfrak{D}$-support of M $= \{\alpha \in \mathfrak{G} \mid \mathfrak{T}_\alpha(M) \neq 0\}$. M is said to have finite $\mathfrak{D}$-support if the $\mathfrak{D}$-support of M is a finite set. Finally, $\mathfrak{T}$ is called finitely $\mathfrak{D}$-decomposable if, whenever $M \in \mathfrak{T}$ such that M has finite $\mathfrak{D}$-support, then $M = \bigoplus_{\alpha \in \mathfrak{G}} \mathfrak{T}_\alpha(M)$.

Theorem 4.24. Let $\mathcal{T}$ be a torsion class with subbasis, P. Let $\{P_\alpha\}_{\alpha \in \mathcal{A}}$ be a partition of P, and let $\mathcal{T}_\alpha$ be the torsion class with subbasis, P_α, for all $\alpha \in \mathcal{A}$. Let $C_\alpha = \{J \subsetneq R \mid R/J$ is I-uniformly negligible for some $I \in P_\alpha\}$. If for all $\alpha \in \mathcal{A}$, $\mathcal{T}_\alpha \cap \bigvee_{\beta \neq \alpha} \mathcal{T}_\beta = 0$, then the following statements are equivalent:

(1) $\mathcal{T}$ is finitely $\mathcal{P}$-decomposable.

(2) $\operatorname{Ext}(R/J, M) = 0$, for all $J \in C_\alpha$ and $M \in \mathcal{T}_{\alpha'}$, where $\alpha \neq \alpha'$.

Proof. (1) ⇒ (2). Consider an exact sequence of the form

$$0 \to M \to X \to R/J \to 0,$$

where $J \in C_\alpha$, and $M \in \mathcal{T}_{\alpha'}$, and $\alpha \neq \alpha'$. Then claim that $\mathcal{T}_\beta(X) = 0$ for all $\beta \notin \{\alpha, \alpha'\}$. Let $x \in \mathcal{T}_\beta(X)$. $(M:x) \in F(\mathcal{T}_\alpha)$, and $(0:x) \in F(\mathcal{T}_\beta)$. Lemma 3.5 implies that $(M:x = (M:x) + (0:x) = R$. Therefore, $x \in M$. This contradicts the fact that $M \cap \mathcal{T}_\beta(X) = 0$, unless $x = 0$. Thus X has finite $\mathcal{P}$-support. Condition (1) implies that the original sequence splits and (2) is established.

(2) ⇒ (1). Let $\rho(M) = \bigoplus_{\alpha \in \mathcal{A}} \mathcal{T}_\alpha(M)$. Lemma 4.10 implies that ρ is a torsion preradical which generates $\mathcal{T}$. Assume that M is a torsion module which has finite $\mathcal{P}$-support. Let $X \in \rho_2(M)$ such that $X + \rho(M) \in \mathcal{T}_{\alpha_0}(M/\rho(M))$. By the construction of $\mathcal{T}_{\alpha_0}$, one may assume, without loss of generality, that $X + \rho(M) \in \rho^I(M/\rho(M))$, for some $I \in P_{\alpha_0}$. Assume that $\alpha_0 \notin \mathcal{P}$-support of M. By condition (2) the exact sequence

$$0 \to \bigoplus_{i=1}^{n} \mathcal{T}_{\alpha_i}(M) \to Rx + \bigoplus_{i=1}^{n} \mathcal{T}_{\alpha_i}(M) \to R/(\bigoplus_{i=1}^{n} \mathcal{T}_{\alpha_i}(M):x) \to 0,$$

where $\{\alpha_1,\ldots,\alpha_n\}$ is the $\mathcal{D}$-support of M, splits. This contradicts the fact that $\mathcal{T}_{\alpha_0}(M) = 0$.

Now assume that $\alpha_0 \in \mathcal{D}$-support of M. Consider the exact sequence

$$0 \to \bigoplus_{\alpha\neq\alpha_0} \mathcal{T}_\alpha(M) \to Rx + \rho(M) \to (Rx+\rho(M)/\bigoplus_{\alpha\neq\alpha_0} \mathcal{T}_\alpha(M) \to 0.$$

The right side of this sequence is in $\mathcal{T}_{\alpha_0}$. However, condition (2), Lemma 4.13, and the fact that M has finite $\mathcal{D}$-support imply that this sequence splits. Therefore, $x \in \mathcal{T}_{\alpha_0}(M) \subseteq \rho(M)$. Consequently, $\rho(M) = \rho_2(M) = \mathcal{T}(M)$ and $\mathcal{T}$ is finitely $\mathcal{D}$-decomposable. □

Corollary 4.25 [1, Theorem 5]. Let $\mathcal{S}$ be the simple torsion class, and let $P = \{M_\alpha\}_{\alpha\in\mathfrak{a}}$ be a separating subbasis for $\mathcal{S}$, where each M_α is a maximal left ideal of R. For all $\alpha \in \mathfrak{a}$, let $\mathcal{T}_\alpha = \mathcal{T}_{R/M_\alpha}$. If $\mathcal{D} = \{\mathcal{T}_\alpha\}_{\alpha\in\mathfrak{a}}$, then the following statements are equivalent:

(1) $\mathcal{S}$ is finitely $\mathcal{D}$-decomposable.

(2) $\mathrm{Ext}(R/M_\alpha, M) = 0$, for all $M \in \mathcal{T}_{\alpha'}$, where $\alpha \neq \alpha'$.

Proof. (1) ⇒ (2). This follows immediately from Theorem 4.24.

(2) ⇒ (1). This follows from Lemma 2.8 and Theorem 4.24. □

CHAPTER 5

Completely Reducible Torsion Classes

Definition 5.1. Let $\mathfrak{T}$ be a torsion class on R-mod. Then $\mathfrak{T}$ is called completely reducible if whenever $\mathfrak{T}_1$ is a torsion class in $\mathfrak{T}$, then there exists a torsion class, $\mathfrak{T}_2$, contained in $\mathfrak{T}$ such that $\mathfrak{T}(M) = \mathfrak{T}_1(M) \oplus \mathfrak{T}_2(M)$ for all $M \in$ R-mod.

The primary goal of this chapter is to characterize those rings for which all torsion classes are completely reducible. These rings have been studied by Dlab [8]. A sequence of lemmas will lead to the main result.

Lemma 5.2. If $\mathfrak{T}$ is a torsion class and if Γ is a complete representative set of non-isomorphic simple modules in $\mathfrak{T}$, then the following statements are equivalent:

(1) $\mathfrak{T}$ is completely reducible.

(2) For each $S_o \in \Gamma$, $\mathfrak{T}(M) = \mathfrak{T}_{S_o}(M) \oplus \bigvee_{S\in\Gamma\setminus\{S_o\}} \mathfrak{T}_S(M)$.

Proof. (1) ⇒ (2). Let $\mathfrak{T}$ be a completely reducible torsion class, and let Γ be a complete representative set of non-isomorphic simple R-modules in $\mathfrak{T}$. Let $\mathfrak{T}_\Gamma$ be the smallest torsion class containing all the simple R-modules in $\mathfrak{T}$. Therefore, $\mathfrak{T}_\Gamma \subseteq \mathfrak{T}$. By hypothesis, there exists a torsion class, $\mathfrak{U}$, such that $\mathfrak{T}(M) = \mathfrak{T}_\Gamma(M) \oplus \mathfrak{U}(M)$ for all $M \in$ R-mod. If $\mathfrak{U}$ were not the zero class, then there would be a non-zero simple R-module,

S, in $\mathcal{U}$. Since $\mathcal{U} \subseteq \mathcal{T}$, $S \in \mathcal{T}$. By the definition of Γ, S is isomorphic to an element of Γ. Therefore, $S \in \mathcal{T}_\Gamma$, which contradicts the fact that $\mathcal{T}_\Gamma \cap \mathcal{U} = \{0\}$. Consequently, $\mathcal{U} = \{0\}$, and $\mathcal{T}$ is of simple type.

Let $S_o \in \Gamma$. By hypothesis, there exists a torsion class, $\mathcal{U}$, such that $\mathcal{T}(M) = \mathcal{U}(M) \oplus \mathcal{T}_{S_1}(M)$ for all $M \in$ R-mod. Since $\mathcal{T}$ is of simple type, $\mathcal{U}$ is of simple type also. Hence, $\mathcal{U} = \bigvee_{S\in\Gamma\setminus\{S_o\}} \mathcal{T}_S$. By Lemma 4.17, $\mathcal{T}$ satisfies the Primary Decomposition.

(2) ⇒ (1). Let $\mathcal{U}$ be a torsion class contained in $\mathcal{T}$, where $\mathcal{T}$ satisfies condition (2). Let $\mathcal{V} = \vee_{S\in\Gamma\setminus\mathcal{U}} \mathcal{T}_S$. $\mathcal{T}(M) = \mathcal{U}(M) \oplus \mathcal{V}(M)$ for all $M \in$ R-mod, because $\mathcal{T}$ satisfies the Primary Decomposition. □

Lemma 5.3. Let R and S be rings with identity, and let $f:R \to S$ be a ring homomorphism. Any S-module, M_1 may be made an R-module via the definition $rm = f(r)m$ for all $r \in R$ and $m \in M$. If $\mathcal{T}$ is a torsion class on R-mod, then $\mathcal{T}^1 = \{M \in S\text{-mod} \mid M \in \mathcal{T}$, if M is considered as an R-module$\}$ is a torsion class on S-mod.

Proof. Recall [5, Theorem 2.3] that a class of modules, $\mathcal{T}$, is a torsion class if $\mathcal{T}$ is closed under submodules, homomorphic images, direct sums, and extensions of one module in $\mathcal{T}$ by another. Noting this fact, the result follows routinely and will be omitted. □

Lemma 5.4. Let $R = S + T$ (ring direct sum), and let $\mathcal{U}$ be a torsion class on S-mod (T-mod). If M is an R-module, then $SM(TM)$ is an S-module (T-module) with the multiplication defined in the obvious way. Then $\mathfrak{U}' = \{M \in R\text{-mod} \mid SM \in \mathfrak{U}\}$ and $\mathfrak{U}'' = \{M \in R\text{-mod} \mid TM \in \mathfrak{U}\}$ are torsion classes on R-mod.

Proof. The proof is routine and will be omitted.

Lemma 5.5. Let $R = R_1 + R_2 + \ldots + R_n$ (ring direct sum). Then the following statements are equivalent:

(1) All torsion classes on R-mod are central splitting.

(2) All torsion classes on R_i-mod are central splitting for $i = 1, 2, \ldots, n$.

Proof. (1) ⇒ (2). It suffices to show that if $\mathfrak{T}$ is a torsion class on R_1-mod, then $\mathfrak{T}$ is central splitting. By Lemma 5.3, $\mathfrak{T}^1 = \{M \in R\text{-mod} \mid R_1 M \in \mathfrak{T}\}$ is a torsion class on R-mod. Since condition (1) implies that $\mathfrak{T}^1$ is central splitting, there exists a torsion class, $\mathfrak{U}^1$ on R-mod such that $M = \mathfrak{T}^1(M) \oplus \mathfrak{U}^1(M)$ for all $M \in R$-mod. In particular, $R = \mathfrak{T}^1(R) \oplus \mathfrak{U}^1(R)$ and $\mathfrak{T}^1 = \{M \in R\text{-mod} \mid \mathfrak{U}^1(R)M = 0\}$. $R_1 = \mathfrak{T}^1(R_1) \oplus \mathfrak{U}^1(R_1)$, and $\mathfrak{T}^1(R_1) = R_1\mathfrak{T}^1(R_1) \in \mathfrak{T}$, because $\mathfrak{T}^1(R_1) \in \mathfrak{T}^1$. Since $\mathfrak{U}^1(R_1) \in F(\mathfrak{T})$, $\{M \in R_1\text{-mod} \mid \mathfrak{U}^1(R_1)M = 0\}$ is contained in $\mathfrak{T}$. Conversely, let $M \in \mathfrak{T}$. Make M an R-module via the definition $rm = (r_1 + \ldots + r_n)m = r_1 m$ for all $m \in M$ and $r \in R$. Since $R_1 M = M \in \mathfrak{T}$, $M \in \mathfrak{T}^1$. Therefore, $\mathfrak{U}^1(R)M = 0$, and consequently, $\mathfrak{U}^1(R_1)M = 0$. Therefore, $\mathfrak{T}$ is central splitting.

(2) ⇒ (1). Let $\mathfrak{T}$ be a torsion class on R-mod, and let f_i be the canonical map from R onto R_i for $i = 1, 2, \ldots, n$. For each $i = 1, 2, \ldots, n$, let $\mathfrak{T}_i$ be the torsion class induced on R_i-mod by f_i, as described in Lemma 5.4. By condition (2) each $\mathfrak{T}_i$ is central splitting. Then each $R_i = \mathfrak{T}_i(R_i) + K_i$ (ring

direct sum), where K_i is the minimal ideal in $F(\mathcal{T}_i)$. Then $R = (\bigoplus_{i=1}^{n} \mathcal{T}_i(R_i)) + \bigoplus_{i=1}^{n} K_i$. Since $\bigoplus_{i=1}^{n} \mathcal{T}_i(R_i) \in \mathcal{T}$, $\bigoplus_{i=1}^{n} K_i \in F(\mathcal{T})$. Therefore, $\{M \in R\text{-mod} \mid (\bigoplus_{i=1}^{n} K_i)M = 0\}$ is contained in $\mathcal{T}$. Conversely, let $M \in \mathcal{T}$. Then $M = \bigoplus_{i=1}^{n} R_iM$. If one makes R_iM into an R_i-module in the obvious way, then $R_iM_i \in \mathcal{T}_i$ for all $i = 1, 2, \ldots, n$. Therefore, R_iM is annihilated by $\bigoplus_{i \neq j} R_j \oplus K_i$ for all i. M is annihilated by the intersection of annihilators, which is $\bigoplus_{i=1}^{n} K_i$. Hence, $\mathcal{T}$ is central splitting. □

We are now ready for the main theorem of this chapter.

Theorem 5.6. For any ring, R, the following statements are equivalent:

(1) All torsion classes on R-mod are central splitting.

(2) R is a finite direct sum (ring direct sum) of full matrix rings over local semi-artinian rings.

(3) All torsion classes on R-mod are completely reducible.

(4) R is semi-artinian and satisfies the Primary Decomposition.

Proof. (2) $\Leftrightarrow$ (4). This is known [8, Theorem 2].

(1) $\Rightarrow$ (2). Let $\mathcal{S}$ be the simple torsion class. Condition (1) implies that $\mathcal{S}$ is central splitting. Hence, there exists a torsion class, $\mathcal{U}$, such that $M = \mathcal{S}(M) \oplus \mathcal{U}(M)$ for all $M \in R$-mod. However, $\mathcal{U}$ cannot contain any simple R-modules, because $\mathcal{S} \cap \mathcal{U} = \{0\}$ and all simple R-modules are in $\mathcal{S}$. Therefore, $\mathcal{U} = \{0\}$, and $\mathcal{S} = R$-mod. Thus, R is semi-artinian.

Let S_o be a non-zero simple R-module, and let $\mathfrak{T}_{S_o}$ be the S_o-primary. Condition (1) implies that $\mathfrak{T}_{S_o}$ is central splitting. Hence, there exists a torsion class, $\mathfrak{U}$, such that $M = \mathfrak{U}_1(M) \oplus \mathfrak{T}_{S_o}(M)$ for all $M \in$ R-mod. In particular, $R = \mathfrak{U}_1(R) \oplus \mathfrak{T}_{S_o}(R)$. If $\mathfrak{T}_{S_o}(R) = 0$, then $R = \mathfrak{U}_1(R)$ and $\mathfrak{T}_{S_o} = \{0\}$. Since $S_o \neq 0$, this is impossible. Therefore, an isomorphic copy of S_o appears in soc(R).

$\mathfrak{U}_1(R)$ and $\mathfrak{T}_{S_o}(R)$ are both two-sided ideals of R, because they are the torsion submodules of R for some torsion class. Hence, their sum is a ring direct sum. First, a short examination of $\mathfrak{T}_{S_o}(R)$ is in order. Lemma 5.5 implies that all torsion classes in $\mathfrak{T}_{S_o}(R)$-mod are central splitting. From the first paragraph of this proof, it follows that $\mathfrak{T}_{S_o}(R)$ is a semi-artinian ring. From the second paragraph one deduces that there is a unique simple $\mathfrak{T}_{S_o}(R)$-module up to isomorphism. Consequently, there are precisely two torsion classes on $\mathfrak{T}_{S_o}(R)$-mod.

Now proceed to the other summand of R; namely $\mathfrak{U}_1(R)$. By Lemma 5.5, every torsion class on $\mathfrak{U}_1(R)$-mod is central splitting. Let S_1 be a simple $\mathfrak{U}_1(R)$-module. Then $\mathfrak{U}_1(R)$ contains an isomorphic copy of S_1. Observe that S_1 can be made into an R-module in a canonical manner, and that S_o is not isomorphic to S_1 as R-modules. By repeating the process described in the preceding paragraph, $\mathfrak{U}_1(R) = \mathfrak{T}_{S_1}(R) \oplus \mathfrak{U}_2(R)$. By making an inductive construction, one obtains that $R = \mathfrak{T}_{S_o}(R) \oplus \mathfrak{T}_{S_1}(R) \oplus \cdots \oplus \mathfrak{T}_{S_n}(R) \oplus \mathfrak{U}_n(R)$, where $\{S_i\}_{i=1,2,\ldots,n}$ is a set of non-isomorphic simple R-modules, and $\mathfrak{U}_n$ is a torsion class on R-mod. Let $I_n = \bigoplus_{i=1}^{n} \mathfrak{T}_{S_i}(R)$, and consider the chain of ideals $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots$. If this chain terminates in a finite number of steps, then $R = \bigoplus_{i=1}^{n} R_i$ (ring direct sum) where each R_i is a ring with only two torsion classes on R_i-mod. However, each R_i is isomorphic to a full matrix ring over a local semi-artinian ring [8, Theorem 2], and condition (2) is established.

Hence, it suffices to show that the chain $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots$ terminates in a finite number of steps. Observe that the chain may be assumed to be strictly increasing. Thus, $J = \bigcup_{n=1}^{\infty} I_n$ is an idempotent two-sided ideal of R. Consequently, $\mathfrak{T}_{R/J} = \{M \in R\text{-mod} \mid JM = 0\}$ is a torsion class of R-mod with J being the minimal ideal in $F(\mathfrak{T}_{R/J})$ [13, Corollary 2.3]. By hypothesis, $\mathfrak{T}_{R/J}$ is central splitting. There exists a torsion class, $\mathfrak{U}$, such that $R = \mathfrak{U}(R) \oplus \mathfrak{T}_{R/J}(R)$, where $J = \mathfrak{U}(R)$. Let $1 = e + f$, where $e \in J$ and $f \in \mathfrak{T}_{R/J}(R)$. Since $e \in J$, $e \in I_n$ for some integer n. In this case, $e\mathfrak{T}_{S_{n+p}}(R) = 0$ for all $p \geq 1$. Since e is the identity for J, this implies that the chain terminates in a finite number of steps. This contradicts the assumption that the chain is infinite, and condition (2) is established.

(4) ⇒ (3). Let $\mathfrak{T}$ be a torsion class on R-mod. Since R is semi-artinian, $\mathfrak{T}$ is of simple type. Let Γ be a complete representative set of non-isomorphic simple R-modules in $\mathfrak{T}$.

Then $\mathcal{T} = \bigvee_{S\in\Gamma} \mathcal{T}_S$. Since R satisfies the Primary Decomposition, $\mathcal{T}(M) = \bigoplus_{S\in\Gamma} \mathcal{T}_S(M)$ for all $M \in R$-mod. If $\mathcal{T}^1$ is a torsion class contained in $\mathcal{T}$, then define $\mathcal{U} = \bigvee_{S\in\Gamma\setminus\mathcal{T}^1} \mathcal{T}_S$. $\mathcal{T}(M) = \mathcal{T}^1(M) \oplus \mathcal{U}(M)$ for all $M \in R$-mod. Thus, $\mathcal{T}$ is completely reducible.

(3) ⇒ (4). This follows immediately from Lemma 5.2 by letting $\mathcal{T} = R$-mod.

(4) ⇒ (1). There exist a finite number of simple R-modules, $S_1,\ldots,S_n$, such that $R = \bigoplus_{i=1}^{n} \mathcal{T}_{S_i}(R)$. If S is any simple R-module, then $S \cong S_i$ for some $i = 1, 2, \ldots, n$. Let $\mathcal{T}$ be any torsion class on R-mod. Then $\mathcal{T} = \bigvee_{S_i\in\mathcal{T}} \mathcal{T}_{S_i}$. $\mathcal{T}(R) = \bigoplus_{S_i\in\mathcal{T}} (\mathcal{T}_{S_i}(R))$. Let $\mathcal{U} = \bigvee_{S_i\notin\mathcal{T}} \mathcal{T}_{S_i}$. Then $R = \mathcal{T}(R) \oplus \mathcal{U}(R)$, and $\mathcal{T}$ is central splitting. □

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BIOGRAPHICAL SKETCH

Richard Bronowitz was born in New York City on April 13, 1946. Upon graduating from Miami Norland Senior High School in 1963, he attended the Massachusetts Institute of Technology where he received his B.S. in mathematics in 1967. He then entered the University of Florida Graduate School, Department of Mathematics. On June 14, 1970, he married Carol Linda Stein. They have a daughter, Amy Michelle, born on May 29, 1971.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Mark L. Teply

Mark L. Teply, Chairman
Assistant Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Alexander R. Bednarek
Professor and Chairman of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Thomas T. Bowman

Thomas T. Bowman
Assistant Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Thomas H. Hanson

Thomas H. Hanson
Assistant Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Theodore Landsman
Professor of Education

This dissertation was submitted to the Department of Mathematics in the College of Arts and Sciences and to the Graduate Council, and was accepted as partial fulfillment of the requirements for the degree of Philosophy.

August, 1972

Dean, Graduate School

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